λ_o shows distinct behavior for $p = 0.75$ and $p = 0.25$. The most dramatic distortion occurs for $p = 1$ and $p = 0$. For these cases, we clearly have $\lambda_{\max} = 0$. For $p = 1$ we have $\lambda_o = 0$ (all stations retransmit at each slot, thus collision is repeated forever); but for $p = 0$ we have $\lambda_o = \lambda e^{-\lambda}/[1 + (1 + \lambda)e^{-\lambda}]$. See the appendix for a quick explanation of this last identity.

The numerical methodology consists at each application of our operators, T, R, and $H(1)$ on general functions $f(z)$ 1) to consider only the Taylor expansion of $f(z)$ truncated to rank M, and 2) to truncate the Taylor expansion of the result to rank M. This methodology allows us working on floating vectors of finite length instead of infinite ones. General continuity theorems ensure that our results converge to real ones when $M \to \infty$. $M = 10$ gives enough accuracy (within five digits) and computations generally last less than one minute on Sun 3/50 with Pascal implementation, to obtain λ_o for any given λ with desired accuracy. The only problem is when λ is close to $\lambda_{\max}$ since iterations of the operator T slowly converge at this point, as has already been mentioned in Section IV. The other operators have iterative convergence rate largely independent of λ.

APPENDIX

Lemma 1: For all integers n and m, we have $\lambda_o[m] = \lambda_o[n]$.

Proof: We know that $\lim_{u\to 1}(1-u)S_n(u) = (1-P_n)\lambda_n$. Let $S(z) = \lim_{u\to 1}(1-u)S(z,u)$, it suffices to prove that the function $S(z)$ is colinear with function $1 - P(z)$. Using (8), it can easily be checked that the limiting function $S(z)$ satisfies the functional equation

$$\begin{aligned} S(z) = {} & S(pz+\lambda) + P(pz+\lambda)S(qz+\lambda) \\ & - S(\lambda)((1+P(\lambda))(1+z) + P_z(\lambda)pz)e^{-z} \\ & - S_z(\lambda)z(p+qP(\lambda))e^{-z}. \end{aligned} \tag{A.1}$$

It can also be easily checked that $1 - P(z)$ is a solution of (A.1) since formal replacement in (A.1) gives (4). A glance to the subsection "Resolution for $X(z)$" gives the way for resolving general equations like (A.1), applying the operator $(1-R)^{-1}$. In general such equations have a single solution, thus we tentatively have $S(z) = 1 - P(z)$. Of course, the latter is wrong since it can be checked (intricate) that $\det = 0$ (in fact this identity is simply guessed since we *know* that $S(z)$ cannot be equal to $1 - P(z)$, since obviously $\lambda_o \neq 1$), therefore we only have colinearity between $S(z)$ and $1 - P(z)$. In other words, equation (A.1) does not suffice for exactly determining $S(z)$ and λ_o.

Expression of λ_o when $p = 0$: When $p = 0$ stations in collision at their first transmission will experience collision at each time they will be allowed to transmit. Therefore, successful transmission can only occur at first attempt. When several stations collides they automatically reach level 1 in the stack on the next slot. Therefore, this next slot only sees the transmission attempt of the packet generated on this slot (thus Poisson of parameter λ). Thus, a success slot follows a collision slot with probability $\lambda e^{-\lambda}$ and an empty slot follows a collision slot with probability $e^{-\lambda}$. A collision slot follows a collision slot with probability $1 - (1+\lambda)e^{-\lambda}$. A collision slot follows a non collision slot with probability 1 because each level of the stack, except level zero, contains more than two stations, and after a non collision level one goes to transmission.

These considerations imply that collision slots occur with frequency $1/[1 + (1+\lambda)e^{-\lambda}]$, empty slots with frequency $e^{-\lambda}/[1 + (1+\lambda)e^{-\lambda}]$, and success slots with frequency $\lambda e^{-\lambda}/[1 + (1+\lambda)e^{-\lambda}]$. Note that all these results are derivable, through more sophisticated paths, from the identities obtained by the general analysis developed in the body of the correspondence.

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Addendum to "Successive Refinement of Information"

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V. Koshelev has brought to our attention the existence of a conference proceedings [1] and two subsequent published papers [2], [3] that should be included in our above paper.[1] Koshelev, indeed, defined the problem of divisibility, which we call successive refinement, many years ago, and argued that a Markovian relationship of the solutions is sufficient to achieve points on the rate-distortion curve. Our paper proves necessity as well and provides an example

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showing that there are indeed rate-distortion problems for which a Markovian relationship does not exist, and therefore for which successive refinement at the rate-distortion limit is impossible.

The upshot of all of this is that the problem of divisibility (successive refinement) of information is solvable at the rate-distortion limit, if and only if the solutions to the rate-distortion problem have a Markov relationship. Sufficiency was established by Koshelev and necessity in the current paper.[1]

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Correction to "The Input–Output Map of a Monotone Discrete-Time Quasireversible Node"

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In the statement of Lemma 3 of the above paper,[1] $\mathcal{M}_S(\lambda)$ and $\mathcal{M}_S(\lambda, \lambda)$ should be replaced by $\cup_{0 \le \eta \le \lambda} \mathcal{M}_S(\eta)$ and $\cup_{0 \le \eta, \zeta \le \lambda} \mathcal{M}_S(\eta, \zeta)$, respectively. The last two sentences of the proof should be replaced by: "By Fatou's lemma, this establishes the compactness of $\cup_{0 \le \eta \le \lambda} \mathcal{M}_S(\eta)$. A similar argument works for $\cup_{0 \le \eta, \zeta \le \lambda} \mathcal{M}_S(\eta, \zeta)$."

Likewise, in the third sentence of the proof of Theorem 7, $\mathcal{M}_S(\lambda)$ should be replaced by $\cup_{0 \le \eta \le \lambda} \mathcal{M}_S(\eta)$. In the concluding remarks, K is a subset of $\cup_{0 \le \eta \le \lambda} \mathcal{M}_S(\eta)$ and the definition of $\bar{\rho}$ has to be extended to processes with unequal rates in the obvious way.

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