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Log Optimal Portfolios †

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ABSTRACT

We shall argue that the log optimal portfolio is natural by exhibiting some new properties of it -- long run, short run, and medium run. For long run, we shall provide an ergodic theorem and an asymptotically optimal portfolio. For short run, we shall solve a 2-person zero-sum game in which one investor seeks to outperform another. For medium run, we shall exhibit a sequential portfolio selection algorithm that performs asymptotically as well as if one could look n days into the future. These algorithms share one thing in common -- they are all log optimal.

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1. Introduction.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ denote a random stock market return vector, where X_i is the value of a one unit investment in stock i at the end of the trading day. Typically X_i wll take on values like .95 or 1.01. We require that $\mathbf{X} \geq \mathbf{0}$, i.e., no investment can be made that causes the investor to owe money at the end of the investment period. Let $\mathbf{b}$, $b_i \geq 0$, $\sum_{i=1}^{m} b_i = 1$, denote a *portfolio*, i.e., an allocation of the investor's capital across the investment alternatives. Thus b_i is the proportion of current capital invested in the i-th stock. The capital return S from investment portfolio $\mathbf{b}$ is the random variable

$$S = \sum b_i X_i = \mathbf{b}^t \mathbf{X} . \tag{1.1}$$

Our goal is to find the "largest" random variable S. This goal is obviously absurd, since we cannot hope to find S^* such that $P(S^* \geq S) = 1$, for all $S = \mathbf{b}^t \mathbf{X}$.

We shall attack this problem by defining the *log optimal portfolio* $\mathbf{b}^*$ to be the portfolio achieving $\max_{\mathbf{b}} E \ln b^t X$.

The correspondence of log optimal portfolios to the mean variance approach investigated by Markowitz, Sharpe and others, is perhaps best described by saying that the first two terms in the Taylor series expansion for the log optimal criterion are proportional to the mean and covariance of the market. Thus the mean-variance theory optimizes the first two terms in the Taylor series expansion of $E \ln b^t X$. However, it should be mentioned that Thorp has found an example of a log optimal portfolio that is not on the efficient frontier in the mean-variance space.

Perhaps it does not need to be said but the log criterion arises, not from a utility theory, but from the strong law of large numbers for products. Just as the expected value of X describes the dominant behavior of $(1/n) \sum_{i=1}^{n} X_i$, so does the expected value of $\log X$ describe the dominant behavior of $\prod X_i$. We have $\frac{1}{n} \sum X_i \to EX$, a.e., and $(\prod X_i)^{1/n} \to e^{E \ln X}$, a.e.

Because of the inequality of the geometric and arithmetic means, we can conclude that if the market goes nowhere, that is, if the product of the incremental factors for each of the days is one, then the expected value is greater than one. Any portfolio scheme that keeps a fixed amount of money in cash and the rest in the market will then make money off the fluctuations. We exhibit in Section 5 a sequential portfolio selection algorithm based solely on the past which not only lives off the fluctuations but follows the drift as well. In fact it does as well (up to first order terms in the exponent) as if the stocks had all been independently identically distributed according to the n-day empirical distribution of the market. This is a result that holds for every sample sequence $x_1, x_2, x_3 \cdots \in \mathbf{X} \subseteq \mathbf{R}_+^m$. Thus bull markets and bear markets can not fool the investor into over-committing or under-committing his capital to the risky alternatives available to him. This goal is accomplished by a game theoretic choice of portfolio which is robust with respect to futures that may differ drastically from the past.

Although it is not surprising to see the log optimal portfolio perform well asymptotically (again because of the strong law of large numbers for products) it is remarkable to see the log optimal portfolio arise as the solution to the one shot game-theoretic competition in which one investor wishes to outperform another. The secret to obtaining this result, given in Section 3, is to allow the right form of convexification, in which fair independent randomization of capital is allowed for each of the players. We are still not sure whether the concidence of solutions for the long and short run problems is an accident of the mathematics or a consequence of some deeper underlying reason.

It is indeed gratifying that the study of ergodic stock markets (Section 6) parallels rather precisely the study of the entropy for ergodic processes pursued by Shannon, McMillan, Breiman, Kolmogorov and Ornstein. The algebra for expected log return $W(X_1, X_2, \ldots, X_n)$ is similar to that for the entropy $H(X_1, \ldots, X_n)$. Also, there is a doubling rate W^* for a stationary market just as there is an entropy rate H for a stationary process. There is an asymptotic equipartition theorem for both. Finally, there are sequential portfolio algorithms achieving a doubling rate W^* ; in fact, it can be shown with probability one that no other portfolio algorithm is asymptotically better than the conditional log optimal portfolio.

In conclusion, we see that the high performance of the log optimal portfolio in the study of Grauer and Hakansson (1982) of the market from 1936 through 1978 is no fluke. It is not surprising that of the half dozen schemes they considered, the log optimal portfolio was essentially tied for first.

In Section 3, we show that $\mathbf{b}^*$ is game theoretically optimal for a single investment. In Section 5 we show, if $\mathbf{b}^*$ is conditionally log optimal with respect to an appropriately modified empirical distribution of the past performance, then the resulting capital outperforms the best stock, to first order in the exponent. Finally, in Section 6, we show that the conditionally log optimal portfolio $\mathbf{b}_n^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-1})$ is asymptotically optimal for ergodic markets.

2. Properties of the log-optimal strategy $\mathbf{b}^*$.

Let

$$W(\mathbf{b},F) = \underset{F}{E} \ln \mathbf{b}^t\mathbf{X} = \int \ln \mathbf{b}^t\mathbf{X}\, dF(\mathbf{x}) . \tag{2.1}$$

denote the expected log return for portfolio $\mathbf{b}$ and distribution F. We shall use the following properties throughout the paper.

Lemma 1: $W(\mathbf{b},F)$ is concave in $\mathbf{b}$ and linear in F.

Proof: Jensen's inquality.

Lemma 2: If $W(\mathbf{b}^*,F) = \max_{\mathbf{b}} W(\mathbf{b},F)$, then

$$E \frac{X_i}{\mathbf{b}^{*t}\mathbf{X}} = 1 , \quad \text{for} \quad b_i^* > 0 \tag{2.2}$$

$$\leq 1 , \quad \text{for} \quad b_i^* = 0 .$$

Proof: These are the Kuhn-Tucker conditions. See Bell-Cover (1980), or Finkelstein and Whitley (1981).

Lemma 3: The set B^* of all log optimal portfolios is a convex set.

Proof: Breiman (1961). See also Bell-Cover (1980), Cover (1984).

3. The game-theoretic market investor.

Suppose one investor wishes to outperform another during a given investment period. Both have access to the known distribution $F(\mathbf{x})$ of the market. What portfolios should each choose?

In a previous paper (Bell-Cover, 1980) we considered a short term goal by solving for the optimal portfolio in a two-person zero-sum game with payoff function $P(S_1 > S_2)$. Of course by symmetry, this game has value $1/2$. Remarkably, the optimal portfolio distribution (after game theoretic randomization) is the log optimal portfolio $\mathbf{b}^*$.

In this section, we generalize this result to games with the more general payoff function $E\ \varphi(S_1/S_2)$. The previous result follows when we let φ be the indicator function of $[1,\infty)$. In the above game theoretic sense, we can argue that portfolio $\mathbf{b}^*$ results in the largest random variable S^*.

Consider the following two-person zero-sum game. Players 1 and 2 each start with one unit of capital. A strategy for player i consists of a choice of a "fair" distribution function $G_i(w)$, $G_i(0^-) = 0$, $\int w dG_i(w) \le 1$, and a choice of portfolio $\mathbf{b}_i \in B$. A player then exchanges his unit of capital for the fair r.v. $W_i \sim G_i(w)$, and distributes the result W_i of this gamble across the stocks according to portfolio $\mathbf{b}_i$. We assume that W_1 and W_2 are independent r.v.'s. The payoff to player 1 for the game is

$$E\ \varphi(W_1\, \mathbf{b}_1^t \mathbf{X}\ /\ W_2\, \mathbf{b}_2^t \mathbf{X}) =$$

$$\int \varphi(w_1\, \mathbf{b}_1^t\, \mathbf{x}\ /\ w_2\, \mathbf{b}_2^t\, \mathbf{x}) dG_1(w_1) dG_2(w_2) dF(\mathbf{x})\ . \tag{3.1}$$

We call this the *stock market φ-game*.

Let $W_1^* \sim G_1^*$, $W_2^* \sim G_2^*$ denote the minimax strategies for the primitive φ—game, and let $v(\varphi)$ denote the value of this game given by

$$v(\varphi) = \inf_{W_2 \in \mathbf{W}} \sup_{W_1 \in \mathbf{W}} E\, \varphi(W_1 / W_2) . \tag{3.2}$$

Let $\mathbf{b}^*$ maximize $E \ln \mathbf{b}^t \mathbf{X}$, and let

$$S^* = \mathbf{b}^{*t} \mathbf{X} . \tag{3.3}$$

Theorem 1: Let $\varphi(t)$ be a monotonic nondecreasing function. Then the two-person zero-sum game with payoff $E\, \varphi(W_1 \mathbf{b}_1^t \mathbf{X} / W_2 \mathbf{b}_2^t \mathbf{X})$ has a value $v(\varphi)$ and optimal strategies

$$W_1^* \sim G_1^* , \quad \mathbf{b}_1^* = \mathbf{b}^* ,$$

$$W_2^* \sim G_2^* , \quad \mathbf{b}_2^* = \mathbf{b}^* . \tag{3.4}$$

Remark: The optimal strategies for the stock market φ-game factor into two parts:

1) the game-theoretic randomization W_1^* , W_2^* , designed solely to win the primitive φ-game when no subsequent market investment is allowed,

and

2) a deterministic choice of portfolio $\mathbf{b}_1^* = \mathbf{b}_2^* = \mathbf{b}^*$,

identical for both players, chosen independently of the payoff criterion φ .

Proof: (Bell, Cover, 1984, paper in preparation).

4. An algorithm for determining the log optimal portfolio.

The problem of maximizing $E \ln \mathbf{b}^t\mathbf{X}$ can be viewed as one of maximizing a concave function over the simplex $B = \{ \mathbf{b} \in \mathbf{R}^m : \mathbf{b} \geq 0, \ \Sigma\, b_i = 1 \}$. Thus a maximizing $\mathbf{b}^*$ exists. Optimization algorithms abound for problems of this kind. This section is based on Cover (1984) and is not crucial to the development of the following sections.

Special properties of the maximization suggest the use of an algorithm specific to the problem. In particular, because of the logarithmic objective function, an algorithm that takes multiplicative rather than additive steps seems natural.

The gradient of $W(\mathbf{b})$, which we denote by $\alpha(\mathbf{b})$, is given by

$$\alpha(\mathbf{b}) = E\, \mathbf{X} / \mathbf{b}^t\mathbf{X} = \nabla W(\mathbf{b}). \tag{4.1}$$

The Algorithm: Generate a sequence of portfolio vectors $\mathbf{b}^n \in B$, recursively according to

$$b_i^{n+1} = b_i^n \alpha_i(\mathbf{b}^n), \quad i = 1,2,\ldots,m, \tag{4.2}$$

$$\mathbf{b}^0 > \mathbf{0}.$$

The algorithm multiplies the current portfolio vector $\mathbf{b}$ by the gradient, component by component. It has the following natural interpretation. Let $\mathbf{b}$ be the current allocation of resources across the stocks. The random vector $\mathbf{X}$ results in current holdings in the ith stock $b_i X_i$ and yields a total return $\mathbf{b}^t\mathbf{X}$. Thus the new proportion of capital in the ith stock is given by $b_i X_i / \mathbf{b}^t\mathbf{X}$, and the expected proportion in the ith stock is

$$b_i' = E(b_i X_i / \mathbf{b}_t\mathbf{X}) = b_i \alpha_i(\mathbf{b}). \tag{4.3}$$

This is the new portfolio induced by the algorithm. One replaces the portfolio $\mathbf{b}$ by the *expected* portfolio $\mathbf{b}'$ induced by one play of the market $\mathbf{X}$.

Naturally one expects that the algorithm terminates at $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{b}$. This is indeed the case.

Let,

$$D(\mathbf{b}'\|\mathbf{b}) = \sum_{i=1}^{m} b_i' \ln (b_i' / b_i) , \tag{4.4}$$

be the Kullback-Leibler information number between $\mathbf{b}'$ and $\mathbf{b}$. See Kullback [14] for extensive interpretations of this definition. It is well known that

$$D(\mathbf{b}'\|\mathbf{b}) \geq 0 , \quad \text{with equality if and only if} \quad \mathbf{b}' = \mathbf{b} , \tag{4.5}$$

a consequence of the strict concavity of the logarithm.

Theorem 2: (Monotonicity) ([10]).

$$W(\mathbf{b}') - W(\mathbf{b}) \geq D(\mathbf{b}' \| \mathbf{b}) \geq 0 . \tag{4.6}$$

Consequently, $W(\mathbf{b}^n)$ is monotonically nondecreasing and thus has a limit. It remains to be shown that the limit is W^*.

Theorem 3: (Convergence).

If $\mathbf{b}^0 > 0$, then

$$W(\mathbf{b}^n) \uparrow W^* . \tag{4.7}$$

Moreover, if $\mathbf{X}$ has full dimension, then $\mathbf{b}^n \to \mathbf{b}^*$.

We now establish upper bounds on the error of approximation of $W(\mathbf{b}^n)$ to W^*. The bound is a function of the portfolio $\mathbf{b}$ and not of the algorithm used to guess $\mathbf{b}$.

Theorem 4: (Upper Bound). For any $\mathbf{b} \in B$,

$$W(\mathbf{b}) \leq W^* \leq W(\mathbf{b}) + \max_i \ln E \frac{X_i}{\mathbf{b}^t \mathbf{X}} . \tag{4.8}$$

Remark: Note that the Kuhn-Tucker conditions require that the term $max \ln E X_i / S(\mathbf{b})$ be ≤ 0 for $W(\mathbf{b}) = W^*$. Thus the upper bound converges to W^* as $n \to \infty$.

5. Convergence to Empirical Bayes Profit in Repeated Play Against the Stock Market.

We now consider sequential investments in a stock market with the goal of performing as well as if we knew the empirical distribution of future market peformance. The work in this section is joint with David Gluss.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \geq \mathbf{0}$ denote a market vector for one investment period, where x_i is the number of units returned from an investment of 1 unit in the i-th stock. A portfolio $\mathbf{b} = (b_1, b_2, \ldots, b_m)$, $\sum b_i = 1$, is the proportion of the current capital invested in each of the m stocks. Thus $S = \mathbf{b}^t \mathbf{x} = \sum b_i x_i$ is the factor by which the capital is increased in one investment period using portfolio $\mathbf{b}$.

For the moment let $\mathbf{X}^1, \mathbf{X}^2, \ldots$ be independent identically distributed random vectors drawn according to $F(\mathbf{x}), \mathbf{x} \in \mathbf{R}^m$, where F is some known distribution function. Let $W(\mathbf{b}, F) = \int \ln \mathbf{b}^t \mathbf{x} dF(\mathbf{x})$, and let $\mathbf{b}(F)$ maximize $W(\mathbf{b}, F)$ over $\mathbf{b}$. It has been shown that the capital $S_n = \prod_{i=1}^{n} \mathbf{b}^t \mathbf{X}^i$ resulting from repeated use of $\mathbf{b}(F)$ results in $\frac{1}{n} \ln S_n \to W(\mathbf{b}(F), F)$ with probability one, and that no higher limit is achievable with probability $\varepsilon > 0$ by any other $\mathbf{b}$.

Now suppose that the stock vectors are not random, but it is known that $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n$ lie in some finite set $\mathbf{X} \subseteq \mathbf{R}^m$. Let F_k denote the empirical cumulative probability distribution function induced by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Thus F_k corresponds to a uniform distribution with mass $\frac{1}{k}$ on each of the points $\mathbf{x}_i$. If the portfolio $\mathbf{b}^k = \mathbf{b}$ is used for each investment period, the resulting capital is given by

$$S_n = \prod_{i=1}^{n} \mathbf{b}^t \mathbf{x}_i$$

$$= e^{n(\frac{1}{n}\sum_{i=1}^{n} \ln \mathbf{b}^t \mathbf{x}_i)}$$

$$= e^{nW(\mathbf{b},F_n)}$$

$$\leq e^{nW(\mathbf{b}(F_n),F_n)} \tag{5.1}$$

We are thus motivated to find a sequential portfolio selection algorithm that achieves $W(\mathbf{b}(F_n),F_n)$. Here $W(\mathbf{b}(F_n),F_n)$ represents the maximal expected log return over all time-invariant portfolios $\mathbf{b}$. In general we cannot hope to achieve $W(\mathbf{b}(F_n),F_n)$, since F_n is not known during the sequence of n investments.

Theorem 5: (Gluss and Cover, 1983, paper in preparation). There exists a sequence of portfolios $\mathbf{b}^k$, where $\mathbf{b}^k$ depends only on $\mathbf{x}_1,\mathbf{x}_2, \ldots, \mathbf{x}_{k-1}$ and the cardinality of $\mathbf{X}$, such that the cumulative log return satisfies

$$\frac{1}{n}\sum_{k=1}^{n} \ln \mathbf{b}^{k_t}\, \mathbf{x}_k \geq W(\mathbf{b}(F_n),F_n) - \frac{c}{\sqrt{n}}, \tag{5.2}$$

for all $\mathbf{x}_1,\mathbf{x}_2, \ldots, \mathbf{x}_n$, and for all n, where the constant c depends only on the range $\mathbf{X}$. Thus one can perform asymptotically as well on sequential investments as if one knew F_n ahead of time.

In particular, we can compare this result to what could be achieved by looking at the newspaper n investment periods in the future. By comparing stock prices then and now, we could determine the stock that increased the most and invest all of our capital in that stock. If we leave that investment untouched, the resulting capital $\tilde{S}_n$ is given by

$$\tilde{S}_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \max_{i=1,2,\ldots,m} x_{i1}x_{i2}\cdots x_{in}$$

$$= e^{n(\max_i \frac{1}{n}\sum_{k=1}^{n} \ln x_{ik})}$$

$$= e^{n \max_{\mathbf{b}\in B_0} W(\mathbf{b},F_n)} , \tag{5.3}$$

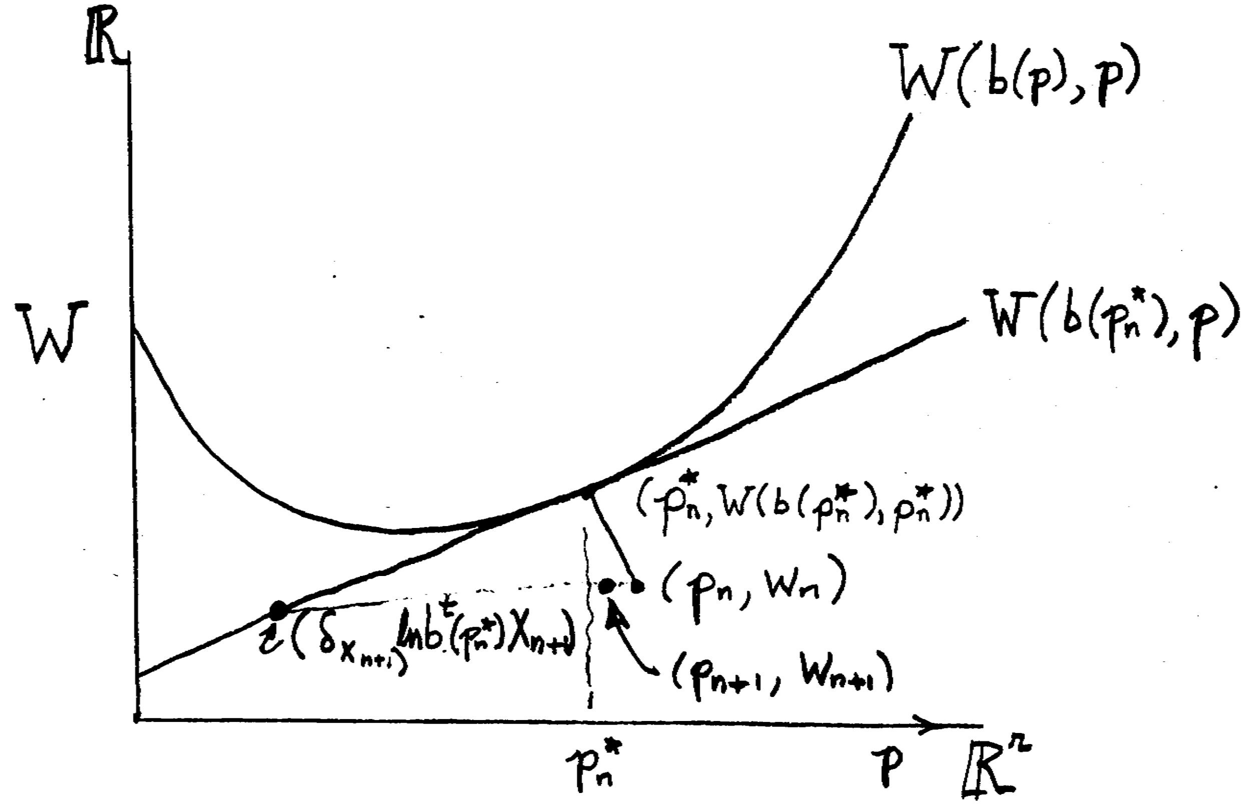
where B_0 is the set of portfolios $B_0 = \{ \mathbf{b} \in B : b_i = 1 \text{ , some } i \}$. Thus, since $B_0 \subseteq B$, the clairvoyant investor achieves capital

$$\tilde{S}_n(x_1, x_2, \ldots, x_n) \le e^{nW(\mathbf{b}(F_n),F_n)} . \tag{5.4}$$

Consequently, $\frac{1}{n}\ln \tilde{S}_n$ is always less than and usually substantially less than the asymptotically achievable goal $W(\mathbf{b}(F_n),F_n)$. Thus the sequential portfolio algorithm outperforms an investor who has knowledge of the far future.

Proof of Theorem 5: (Outline). Consider the choice of portfolio $b^{(n+1)}$ at time n. Let the current capital be $S_n = \prod_{k=1}^{n} \mathbf{b}^{(k)t}\mathbf{X}_k$, with associated empirical average log return $W_n = \frac{1}{n}\ln S_n$. Finally, let $p_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_x(x_i)$ denote the empirical probability mass function on $\mathbf{X}$, where $I_x(x') = 1$, $x' = x$, and $I_x(x') = 0$, otherwise. If the cardinality of $\mathbf{X}$ is r , then p_n lies in an r-dimensional simplex. At time n , we know p_n and W_n. We first determine the closest point $(p_n^*, W(b(p_n^*),p_n^*))$ on the log optimal graph to the point $(p_n, W_n) \in \mathbf{R}^{r+1}$. (See Fig. 1.) Now let $b^{(n+1)} = b(p_n^*)$ be the choice of portfolio for the next investment. It is easily verified that

$$(p_{n+1}, W_{n+1}) = \frac{n}{n+1}(p_n, W_n) + \frac{1}{n+1}(\delta_{\mathbf{X}_{(n+1)}}, \ln \mathbf{b}^{(n+1)t}\mathbf{X}_{n+1}) . \tag{5.5}$$



Log capital vs. the empirical distribution on the stocks

Figure 1.

But the next outcome

$$(\delta_{\mathbf{X}_{n+1}}, \ln \mathbf{b}^{(n+1)t}\, \mathbf{X}_{n+1})$$

falls in the tangent hyperplane

$$(\mathbf{p}, W(\mathbf{b}(\mathbf{p}_n^*), \mathbf{p})), \ \mathbf{p} \geq 0, \ \sum p_i = 1 .$$

Thus for sufficiently large n, $(\mathbf{p}_{n+1}, W_{n+1})$ is closer to the epigraph $(\mathbf{p}, W(\mathbf{b}(\mathbf{p}), \mathbf{p}))$ than was $(\mathbf{p}_n, W_n)$. Careful analysis of this yields the $O(1/\sqrt{n})$ bound given in the statement of the theorem.

6. Asymptotic theory: an asymptotic equipartition theorem for the ergodic stock market.

Work on the asymptotic theory of log optimal portfolios began with Kelly (1956) and Breiman (1961). Breiman proves that if $\mathbf{X}_1, \mathbf{X}_2 \cdots$ are independent identically distributed according to a distribution function $F(\mathbf{x})$ and if the log optimal portfolio $\mathbf{b}^*$ is used at each investment period, then

$$\frac{1}{n} \ln S_n \to E \ln b^t X \,, \quad a.e. \tag{6.1}$$

This is a consequence of the strong law of large numbers. Moreover, it has been shown [13] that S_n / S_n^* converges with probability one to a random variable Y with $E\,Y \leq 1$.

In this section, we shall examine the stock market under the assumption that it is an ergodic process. The results of this section are joint with Paul Algoet (Algoet, Cover, 1983, paper in preparation.) Suppose that $\mathbf{X}_1, \mathbf{X}_2, \ldots$ are m-dimensional nonnegative vector valued observations from a stationary ergodic process. We shall be concerned with sequential portfolio algorithms in which portfolio $\mathbf{b}_1$ is based on no knowledge, $\mathbf{b}_2(\mathbf{X}_1)$ is based on knowledge solely of the previous stock market outcome $\mathbf{X}_1$, and in general, $\mathbf{b}_n(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-1})$ is based on $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$. Thus the capital at time n is given by $S_n = \prod_{k=1}^{n} \mathbf{b}_k^t(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k-1})\mathbf{X}_k$. Assume the distribution of the process is known. In particular, we know the conditional distribution of $\mathbf{X}_n$ given $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-1}$. What now is the optimal portfolio selection policy?

Motivated by this success of the log optimal portfolio for the independent identically distributed stock market case, we will now define the conditionally log optimal portfolio $b_n^*(\mathbf{X}_1, \ldots, \mathbf{X}_{n-1})$ to be the portfolio achieving

$$\max_{\mathbf{b}} E[\ \ln \mathbf{b}^t \mathbf{X}_n \mid \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-1}\ ] . \tag{6.2}$$

It can be shown that the sequential portfolio maximizing $E \ln S_n$ is the conditionally log optimal portfolio defined above. Moreover, if we let $W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)$ be the expectation of the maximum expected log return over all sequential portfolios, we have the following nice chain rule.

Lemma 6.1:

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) =$$

$$W(\mathbf{X}_1) + W(\mathbf{X}_2 | \mathbf{X}_1) + \cdots + W(\mathbf{X}_n | \mathbf{X}_{n-1}, \ldots, \mathbf{X}_1) . \tag{6.3}$$

Remark: This is in precise parallel with the chain rule for entropies for which it is known that the entropy $H(X_1, X_2, \ldots, X_n) = -\int f(\mathbf{x}) \ln f(\mathbf{x})$ satisfies

$$H(X_1, X_2, \ldots, X_n) = \sum H(X_i \mid X_1, \ldots, X_{i-1}) . \tag{6.4}$$

Now, let us assume that $\{ \mathbf{X}_i \}$ is stationary. We have the following lemma.

Lemma 6.2: ([1]). If $\{ \mathbf{X}_i \}$ is a stationary process, then $W(\mathbf{X}_0 | \mathbf{X}_{-1}, \mathbf{X}_{-2}, \ldots, \mathbf{X}_{-n})$ is monotonic nondecreasing and therefore has a limit W^*. Moreover,

$$W^* = \lim W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) / n \tag{6.5}$$

The above lemma suggests that it might be possible to achieve asymptotic capital growth rate W^* in an ergodic stock market. This is indeed true as we state in the following theorem.

Theorem 6: (Algoet, Cover, 1983, paper in preparation). The conditionally log optimal portfolio $b_n^*(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1})$ achieves capital return S_n^* such that with probability one,

$$\frac{1}{n} \ln S_n^* \to W^* . \tag{6.6}$$

Proof: The process $\{ b_k^{*t}(\mathbf{X}_1, \ldots, \mathbf{X}_{k-1})\mathbf{X}_k \}$ is not ergodic, but appropriate use of the Martingale theorem, Birkhoff's ergodic theorem, and a sandwich trick yield the desired result.

Thus it is clear that the probability one growth rate of capital is W^*. Now we wish to prove that no other investment algorithm does better. We first state the following lemma.

Lemma 6.2: ([1]). For any sequential portfolio algorithm, with associated capital S_n,

$$E(S_n \,/\, S_n^*) \le 1 \,. \tag{6.7}$$

Finally, this lemma allows the proof of the following theorem.

Theorem 7: ([1]).

$$\overline{\lim}\, (\frac{1}{n}) \ln (S_n \,/\, S_n^*) \le 0 \,, \quad \text{with probability one.} \tag{6.8}$$

Consequently, the conditional log optimal portfolio for stationary ergodic processes is asymptotically optimal. This extends the work of Breiman on independent processes to parallel results on stationary ergodic market processes.

7. Summary.

The conditionally log optimal portfolio is game-theoretically optimal for markets with arbitrary known time dependence. In fact, if the market is also stationary and ergodic, this sequential portfolio is also asymptotically optimal with probability one. Finally, if the market process is an arbitrary sequence of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots$ from some finite subset of $\mathbf{R}^m$, there is an algorithm that asymptotically achieves the same average log return as if the $\mathbf{x}_i$'s were independently drawn according to the empirical distribution $F_n(\mathbf{x})$ of the sequence.

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