Chapter in *Open Problems in Communication and Computation*. Springer-Verlag, 1987. T. Cover and B. Gopinath, editors.

5.5 LINEAR SEPARABILITY

Thomas M. Cover

Departments of Electrical Engineering
and Statistics
Stanford University
Stanford, CA 94305

Let $(\mathbf{X}_i, \theta_i)$, $i = 1, 2, \ldots, n$, be i.i.d. random pairs, where $\{ \theta_i \}$ is Bernoulli with parameter 1/2, and $\mathbf{X}_i \sim f_{\theta_i}(\mathbf{x})$, $\mathbf{x}_i \in \mathbf{R}^d$. We say $\{ (\mathbf{X}_i, \theta_i) \}_{i=1}^n$ is *linearly separable* if there exits a vector $\mathbf{w} \in \mathbf{R}^d$ and a real number T such that

$$\begin{aligned} \mathbf{w}^t \mathbf{x}_i &\geq T , \quad \theta_i = 1 \\ &< T , \quad \theta_i = 0 , \qquad \text{for} \quad i = 1, 2, \ldots, n . \end{aligned}$$

Let $P(n, d, f_0, f_1)$ be the associated probability that $\{ (X_i, \theta_i) \}_{i=1}^n$ is linearly separable.

The following results are known.

Theorem 1: Identical distributions [1,2].

$$P(n, d, f, f) = 2^{-(n-1)} \sum_{i=0}^{d} \binom{n-1}{i} ,$$

for any density $f(\mathbf{x})$.

Theorem 2: Distributions differing by translation [3].

Let $f_2(\mathbf{x}) = f_1(\mathbf{x} + t\,\mathbf{v})$. Then $P(n, d, f_1, f_2)$ is monotonically increasing in $t \geq 0$. When $t = 0$, $P(n, d, f_1, f_2) = P(n, d, f, f)$, and $P(n, d, f_1, f_2) \rightarrow 1$, as $t \rightarrow \infty$.

Theorem 3:. Distributions differing by scale (Krueger, unpublished).

Let $f_2(x) = \frac{1}{a} f_1(ax)$, $a > 0$. Then $P(n, d, f_1, f_2)$ is monotonically nondecreasing in a , for $a \geq 1$.

All this seems to suggest that different densities lead to an increase in the probability of separability. Hence the following:

Conjecture.

$$P(n, d, f_1, f_2) \geq \left(\frac{1}{2}\right)^{n-1} \sum_{i=0}^{d} \binom{n-1}{i},$$

for all densities $f_1(x), f_2(x)$.

REFERENCES

[1] T. Cover, "Geometrical and Statistical Properties of Systems of Linear Inequalities with Applications in Pattern Recognition," *IEEE Trans. Electronic Comput.*, EC-14, No. 3, pp. 326-334 (June 1965).

[2] R.O. Winder, "Single State Threshold Logic," *Switching Circuit Theory and Logical Design,* AIEE Special Publications S-134, pp. 321-332, September 1961.

[3] F. Bruckstein and T. Cover, "Monotonicity of Linear Separability under Translation," *IEEE Trans. Pattern Anal. Machine Intelligence,* PAMI-7, No. 3, pp. 355-358 (May 1985).