GEOMETRICAL PROBABILITY AND RANDOM POINTS ON A HYPERSPHERE[1]

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0. Summary. This paper is concerned with the properties of convex cones and their dual cones generated by points randomly distributed on the surface of a d-sphere. For radially symmetric distributions on the points, the expected number of k-faces and natural measure of the set of k-faces will be found.

The expected number of vertices, or extreme points, of convex hulls of random points in E^2 and E^3 has been investigated by Rényi and Sulanke [4] and Efron [2]. In general these results depend critically on the distribution of the points. However, for points on a sphere, the situation is much simpler. Except for a requirement of radial symmetry of the distribution on the points, the properties developed in this paper will be distribution-free. (This lack of dependence on the underlying distribution suggests certain simple nonparametric tests for radial symmetry—we shall not pursue this matter here, however.)

Our approach is combinatorial and geometric, involving the systematic description of the partitioning of E^d by N hyperplanes through the origin. After a series of theorems counting the number of faces of cones and their duals, we are led to Theorem 5 and its probabilistic counterpart Theorem 2′, the primary result of this paper, in which the expected solid angle is found of the convex cone spanned by N random vectors in E^d.

1. Introduction. It is known that N hyperplanes in general position in E^d divide E^d into

$$C(N, d) = 2 \sum_{i=0}^{d-1} \binom{N-1}{i} \tag{1.1}$$

regions. (A set of N vectors in Euclidean d-space E^d is said to be in general position if every d-element subset is linearly independent, and a set of N hyperplanes through the origin of E^d is said to be in general position if the corresponding set of normal vectors is in general position.)

This result, first proved[2] by Schläfli [5] in the 19th century, is an intrinsic property of collections of hyperplanes in the sense that the number of nondegenerate cones formed is independent (subject to general position) of the configuration of the normal vectors. Schläfli's theorem, essentially combinatorial

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[2] This theorem was proved in 2 and 3 dimensions in 1826 by Steiner [7] and was later generalized to d dimensions by Schläfli. Although the 2-dimensional case is trivial, the 3-dimensional case has in it the fundamental elements of the multidimensional case. For this reason, Theorem 1 is sometimes called Steiner's theorem.

in nature, has been restated to yield useful results in many branches of mathematics, as the following examples show:

(a) Given $x_1, x_2, \cdots, x_N$ in general position in E^d, consider the set of simultaneous inequalities given by the relations

$$\operatorname{sgn}(x_i \cdot w) = \delta_i, \qquad i = 1, 2, \cdots, N,$$

where sgn is the signum function defined on the reals

$$\begin{aligned} \operatorname{sgn}(y) &= 1, & y > 0, \\ &= 0, & y = 0, \\ &= -1, & y < 0, \end{aligned}$$

and each $\delta_i = \pm 1$. Then among the 2^N possible assignments of the δ_i, exactly $C(N, d)$ will admit some solution vector w. That is, $C(N, d)$ of the 2^N sets of inequalities will be consistent.

(b) Of the 2^N partitions of the vectors $x_1, x_2, \cdots, x_N$ (d-dimensional and in general position) into two subsets, exactly $C(N, d)$ can be separated by a hyperplane through the origin. (A dichotomy is separated by a hyperplane if the two classes lie entirely on opposite sides of the hyperplane.) This formulation of (1.1) is relevant to the theory of linear threshold devices, [1].

(c) Let N vectors be chosen independently according to a d-dimensional probability distribution which is symmetric about the origin ($\mu(A) = \mu(-A)$ for every measurable set A, where $-A = \{x: -x \in A\}$) and absolutely continuous with respect to Lebesgue measure on E^d.[3] Then with probability $C(N, d)/2^N$ there will exist a half-space containing the set of N vectors. This probabilistic formulation of (1.1) is due to Wendel [8].

The $C(N, d)$ regions generated by hyperplanes in general position through the origin of d-space are all proper, nondegenerate convex cones. It is the purpose of this paper to demonstrate other properties of these cones and their dual cones, which, like the number $C(N, d)$, depend on the orientation of the partitioning hyperplanes only through the condition of general position. Applications of invariant properties analogous to (a), (b), and (c) above will be obvious in most cases. When they are not, or when the result is deemed of independent interest, they will be stated explicitly.

2. Theorems and proofs. Let $x_1, x_2, \cdots, x_N$ be a set of N vectors in general position in Euclidean d-space, and let $H_1, H_2, \cdots, H_N$ be the N corresponding hyperplanes through the origin:

$$(2.1) \qquad H_i = \{w : w \cdot x_i = 0\}, \qquad i = 1, 2, \cdots, N.$$

The N hyperplanes partition E^d into $C(N, d)$ proper, nondegenerate (of full dimension) cones W_j, $j = 1, 2, \cdots, C(N, d)$, where $C(N, d)$ is given by (1.1).

The interior of each cone W_j is the set of all solution vectors w to a certain

[3] These conditions can be weakened. See Section 3.

set of simultaneous linear inequalities given by the relations

$$\operatorname{sgn}(x_i \cdot w) = \delta_i, \qquad i = 1, 2, \cdots, N, \tag{2.2}$$

where each $\delta_i = \pm 1$. (Of the 2^N possible vectors of ± 1's, $\delta = (\delta_1, \delta_2, \cdots, \delta_N)$, exactly $C(N, d)$ yield consistent inequalities and hence non-empty solution cones.)

The boundary of the d-dimensional solution cone W_j is the union of a finite number of $(d - 1)$-dimensional cones, which will be referred to as the $(d - 1)$-faces of W_j. The boundaries of the $(d - 1)$-faces are in turn composed of $(d - 2)$-dimensional cones, the $(d - 2)$-faces of W_j. In general, the k-faces of W_j will be proper cones contained in a k-dimensional but not $(k - 1)$-dimensional subspace of E^d. The 1-faces are the extreme rays of W_j, while the origin is the only 0-face. In the following, k will always satisfy $1 \leqq k \leqq d - 1$.

The interior (relative to the smallest subspace containing it) of each k-face of W_j is the totality of solutions to some set of simultaneous relations

$$\operatorname{sgn}(x_i \cdot w) = \delta_i^*, \qquad i = 1, 2, \cdots, N, \tag{2.3}$$

where I is a subset of size $d - k$ of the integers $\{1, 2, \cdots, N\}$ and

$$\begin{aligned} \delta_i^* &= 0, \qquad i \in I, \\ \delta_i^* &= \delta_i, \qquad i \notin I. \end{aligned} \tag{2.4}$$

THEOREM 1. (Counting the k-faces of the solution cones). *Let $R_k(W_j)$ be the number of k-faces of the cone W_j, $j = 1, 2, \cdots, C(N, d)$. Then*

$$\sum_{j=1}^{C(N,d)} R_k(W_j) = 2^{d-k} \binom{N}{d-k} C(N - d + k, k). \tag{2.5}$$

PROOF. Let $H = \bigcap_{i=1}^{d-k} H_i$ be the k-dimensional linear subspace orthogonal to the vectors $x_1, x_2, \cdots, x_{d-k}$. The remaining $N - d + k$ hyperplanes H_{d-k+1}, $H_{d-k+2}, \cdots, H_N$ partition H into $C(N - d + k, k)$ convex cones $\{V_l\}$. (This is easily verified by noting that the projections of $x_{d-k+1}, x_{d-k+2}, \cdots, x_N$ into H are in general position in that space, and that the intersection of H_i with H, for $d - k < i \leqq N$, is the $(k - 1)$-dimensional subspace of H orthogonal to the projection of x_i. Hence the result (1.1) applies.)

The interiors (in H) of each of the cones V_l can be characterized as the set of solution vectors to the simultaneous relations

$$\operatorname{sgn}(x_i \cdot w) = \delta_i^*, \qquad i = 1, 2, \cdots, N, \tag{2.6}$$

where

$$\delta_i^* = 0, \qquad i = 1, 2, \cdots, d - k,$$

and

$$\delta_i^* = \pm 1, \qquad i = d - k + 1, \cdots, N.$$

Let $\delta = (\delta_1, \cdots, \delta_N)$ be a vector of ± 1's such that $\delta_i = \delta_i^*$ for $i \geqq d - k + 1$.

It follows by continuity that every such δ represents (as in (2.2) and (2.3)) a nonempty solution cone having V_l as a k-boundary, and that these 2^{d-k} solution cones are the only ones having this property.

Finally, any of the $\binom{N}{d-k}$ subsets of the size $d - k$ from $x_1, \cdots, x_N$ may be used in place of $x_1, x_2, \cdots, x_{d-k}$ in the discussion above, yielding a total of $\binom{N}{d-k} 2^{d-k}C(N - d + k, k)$ k-boundaries for the solution cones.

To each choice of $d - k$ vectors $x_{i_1}, x_{i_2}, \cdots, x_{i_{d-k}}$ from the set $\{x_1, x_2, \cdots, x_N\}$, there corresponds a k-dimensional orthogonal subspace $L_k(i)$. These subspaces are distinct because of the condition of general position. The proof of Theorem 1 provides some obvious but useful additional information on the k-faces of the solution cones, which is summarized in Theorem 2.

THEOREM 2. *Each k-face of a solution cone W_j is contained in exactly one $L_k(i)$, and the union of the k-faces of all the W_j is the set formed by the union of the $\binom{N}{d-k}$ subspaces $L_k(i)$. Each k-face bounds exactly 2^{d-k} solution cones.*

Given any convex cone W; the *dual cone* W^* is defined to be the set of vectors within a right angle of every vector in W; thus $W^* = \{w^*: w^* \cdot w \geqq 0$ for all $w \in W\}$. In particular, if W is the solution cone corresponding to the set of linear inequalities

$$\text{(2.7)} \qquad \operatorname{sgn}(x_i \cdot w) = \delta_i, \qquad \delta_i = \pm 1, \qquad i = 1, 2, \cdots, N,$$

then it is known that the dual cone W^* is given by

$$\text{(2.8)} \qquad W^* = \{w^*: w^* = \textstyle\sum_{i=1}^{N} \alpha_i \delta_i x_i, \quad \alpha_i \geqq 0, \qquad i = 1, 2, \cdots, N\}.$$

That is, W^* is the proper convex cone *spanned* by the vectors $\delta_1 x_1, \delta_2 x_2, \cdots, \delta_N x_N$. (The $2^N - C(N,d)$ sets of assignments of the δ_i's which lead to an inconsistent set of inequalities (2.7), generate *improper* cones in (2.8).)

As has been shown above, a k-face of the solution cone W_j is orthogonal to exactly $d - k$ of the vectors x_{i_j}, say $x_{i_1}, x_{i_2}, \cdots, x_{i_{d-k}}$. In the (d − k)-dimensional subspace generated by these vectors

$$\text{(2.9)} \qquad L_{d-k}^* = \{x: x = \textstyle\sum_{m=1}^{d-k} c_{i_m} x_{i_m}\},$$

there is one $(d - k)$-face of the dual cone W_j^*—namely, the face spanned by the vectors $\delta_{i_1} x_{i_1}, \delta_{i_2} x_{i_2}, \cdots, \delta_{i_{d-k}} x_{i_{d-k}}$. Thus there is a one-to-one correspondence between the k-faces of a solution cone W_j and the $(d - k)$-faces of its dual cone W_j^*. Immediately, from Theorem 1, we obtain

THEOREM 3. (Counting the k-faces of the dual cones). *Let $R_k(W_j^*)$ be the number of k-faces of the dual cone W_j^*. Then*

$$\text{(2.10)} \qquad \textstyle\sum_{j=1}^{C(N,d)} R_k(W_j^*) = 2^k \binom{N}{k} C(N - k, d - k),$$

$$k = 1, 2, \cdots, d - 1.$$

A statement corresponding to Theorem 2 can also be made for the dual cones. Let $\{L_k^*(1), L_k^*(2), \cdots, L_k^*(\binom{N}{k})\}$ represent the class of k-dimensional linear subspaces of E^d generated by the $\binom{N}{k}$ possible k-element subsets of the N vectors $x_1, x_2, \cdots, x_N$.

THEOREM 4. *Each k-face of a dual cone W_j^* is contained in exactly one $L_k^*(i)$, and the union of the k-faces of all the W_j^* is the set formed by the union of the $\binom{N}{k}$ subspaces $L_k^*(i)$. Each k-face bounds exactly $C(N - k, d - k)$ dual cones. The $(k - 1)$-dimensional interiors of the k-faces overlap only if the two k-faces are identical.*

PROOF. Each of the 2^k cones generated by the vectors $\{\delta_1 x_1, \delta_2 x_2, \cdots, \delta_k x_k\}$, $\delta_i = \pm 1$, is a proper cone, and these cones partition the linear space L_k^* generated by $x_1, x_2, \cdots, x_k$. (The cones overlap only on their boundaries, not on their interiors.)

Let V_k^* be the cone generated by $x_1, x_2, \cdots, x_k$. V_k^* will be a k-face of the convex cone W^* generated by $\{x_1, x_2, \cdots, x_k, \delta_{k+1}x_{k+1}, \delta_{k+2}x_{k+2}, \cdots, \delta_N x_N\}$ if and only if the projections of the vectors $\delta_i x_i$, $i = k + 1, k + 2, \cdots, N$, into L_{d-k}, the orthocomplement of L_k^*, generate a *proper* convex cone in that space. (For, as mentioned previously, V_k^* is a k-face of W^* if and only if it corresponds to a $(d - k)$-face V_{d-k} of its dual cone W. If so any vector w within V_{d-k} will lie in L_{d-k} and will satisfy sgn $(\hat{x}_i \cdot w) = \delta_i$, where $\hat{x}_i$ is the projection of x_i into L_{d-k}, $i = k + 1, \cdots, N$. Conversely, the existence of such a w easily implies W^* is proper and V_k^* is on its boundary.) By Schläfli's theorem exactly $C(N - k, d - k)$ assignments of the signs δ_i, $i = k + 1, \cdots, N$, will have this property. (Note that the projected vectors will be in general position in the $(d - k)$-dimensional space L_{d-k}.)

Thus the k-faces of dual cones partition the subspace L_k^* into 2^k cones, and each k-face bounds $C(N - k, d - k)$ different dual cones. Repeating this argument for the $\binom{N}{k}$ possible selections of k vectors from $x_1, x_2, \cdots, x_N$ completes the proof.

A separate argument is required to establish the next theorem, application of which will yield the expected volume of the cone spanned by a random collection of vectors.

Let $W^*(x_1, x_2, \cdots, x_N)$ denote the convex cone spanned by $x_1, x_2, \cdots, x_N$, and consider the 2^N cones $W^*(\delta_1 x_1, \delta_2 x_2, \cdots, \delta_N x_N)$ where $\delta_1 = \pm 1$, $i = 1, 2, \cdots, N$. Clearly, for $N = d$, this collection of 2^d cones partitions E^d. We shall now show that for $N > d$ the cones $W^*(\delta_1 x_1, \cdots, \delta_N x_N)$ partition E^d $\binom{N-1}{d-1}$ times over in a systematic manner.

THEOREM 5. *Let $x_1, x_2, \cdots, x_N$ lie in general position in E^d. If v is a point in E^d such that $x_1, x_2, \cdots, x_N$ and v jointly lie in general position, then v is a member of precisely $\binom{N-1}{d-1}$ proper convex cones of the form $W^*(\delta_1 x_1, \delta_2 x_2, \cdots, \delta_N x_N) = \{w : w = \sum_{i=1}^N \alpha_i \delta_i x_i, \alpha_i \geqq 0\}$, $\delta_i = \pm 1$, $i = 1, 2, \cdots, N$.*

PROOF. Let $W(x_1, x_2, \cdots, x_N)$ be defined to be the intersection of the half-spaces $\bigcap_{i=1}^N \{w : w \cdot x_i > 0\}$. Let v partition the set S of cones $W(\delta_1 x_1, \cdots, \delta_N x_N)$, the dual cones to the cones $W^*(\delta_1 x_1, \cdots, \delta_N x_N)$, $\delta_i = \pm 1$, $i = 1, 2, \cdots, N$, into three sets defined by

$$
\begin{aligned}
S^+ &= \{W \varepsilon S : v \cdot w > 0, \quad \text{all} \quad w \varepsilon W\}, \\
S^0 &= \{W \varepsilon S : v \cdot w = 0, \quad \text{some} \quad w \varepsilon W\}, \qquad (2.11) \\
S^- &= \{W \varepsilon S : v \cdot w < 0, \quad \text{all} \quad w \varepsilon W\}.
\end{aligned}
$$

There are $C(N, d)$ non-empty cones in S; and there are $C(N, d - 1)$ non-empty cones in S^0 by Schläfli's theorem applied to the projections of the vectors x_i into the space orthogonal to v. Since S^- is the set of reflected cones of S^+, the number of elements in S^+ and S^- is equal, and thus the number of elements in S^+ is

$$\tfrac{1}{2}(C(N, d) - C(N, d - 1)) = \binom{N-1}{d-1}. \tag{2.12}$$

Finally, by the duality of W and W^*, $v \in W^*(\delta_1 x_1, \delta_2 x_2, \cdots, \delta_N x_N)$ if and only if $W(\delta_1 x_1, \delta_2 x_2, \cdots, \delta_N x_N)$ is in S^+.

3. Applications to geometrical probability. Let $X_1, X_2, \cdots, X_N$ be N random points in E^d having a joint distribution invariant under reflections through the origin—that is, for any N sets $A_1, A_2, \cdots, A_N$ in E^d, the probability $P(\delta_1 X_1 \in A_1, \delta_2 X_2 \in A_2, \cdots, \delta_N X_N \in A_N)$ has the same value for all 2^N choices of $\delta_i = \pm 1$. (Actually, as will be clear, it is sufficient for the symmetry condition to hold for all *cones* $A_1, A_2, \cdots, A_N$ in E^d.) Furthermore, let us suppose that with probability one the set of points is in general position. (This is satisfied in the important case where the X_i are selected independently according to a distribution absolutely continuous with respect to natural Lebesgue measure.)

Wendel utilizes Schläfli's theorem in the following manner to establish result (c) of the introduction. Given that $X_1 = \delta_1 x_1, X_2 = \delta_2 x_2, \cdots, X_N = \delta_N x_N$ for some fixed set of points $x_1, x_2, \cdots, x_N$, the symmetry condition implies that all 2^N choices of $\delta_i = \pm 1$ are equally likely; and by Schläfli's theorem, for exactly $C(N, d)$ of these choices the vectors $\delta_1 x_1, \delta_2 x_2, \cdots, \delta_N x_N$ will generate a proper convex cone. The probability that $X_1, X_2, \cdots, X_N$ all lie in some half-space of E^d, or that the unit vectors along the X_i all lie in some one hemisphere of the unit d-sphere, is therefore $C(N, d)/2^N$.

This same argument yields probabilistic statements of Theorems 1 and 3:

THEOREM 1′. *Let W be the random polyhedral convex cone resulting from the intersection of N random half-spaces in E^d with positive normal vectors $X_1, X_2, \cdots, X_N$ having a joint distribution as described above. Then the expected number of k-faces $R_k(W)$ of W, conditioned on $W \neq \Phi$, is given by*

$$E\{R_k(W)\} = 2^{d-k}\binom{N}{d-k} C(N - d + k)/(CN, d) \tag{3.1}$$

and

$$\lim_{N\to\infty} E\{R_k(W)\} = 2^{d-k}\binom{d-1}{d-k}. \tag{3.2}$$

THEOREM 3′. *Let W^* be the random polyhedral convex cone spanned by the collection of random vectors $X_1, X_2, \cdots, X_N$. Then the expected number of k-faces of W^*, conditioned on W^* being a proper cone, is given by*

$$E\{R_k(W^*)\} = 2^k\binom{N}{k} C(N - k, d - k)/C(N, d), \tag{3.3}$$

and

$$\lim_{N\to\infty} E\{R_k(W^*)\} = 2^k\binom{d-1}{k}. \tag{3.4}$$

(Note: by Wendel's result, $P(W \neq \Phi) = P(W^* \text{ proper}) = C(N, d)/2^N$.)

Let μ be any probability measure absolutely continuous with respect to natural Lebesgue measure.[4]

THEOREM 2′. *The expected μ-measure of a non-empty random W described in Theorem 1′ is $1/C(N, d)$. The expected μ-measure of a proper random cone W^* spanned by the collection of random vectors $X_1, X_2, \cdots, X_N$, is $\binom{N-1}{d-1}/C(N, d)$.*

PROOF. Given that $X_1 = \delta_1 x_1, \cdots, X_N = \delta_N x_N$, the $C(N, d)$ non-empty cones W_j, generated (as in Theorem 1′) by different choices of the $\delta_i = \pm 1$, partition E^d, ignoring their boundaries, which have μ-measure 0. Therefore $\sum_{j=1}^{C(N,d)} \mu(W_j) = 1$, and $EW = [C(N, d)]^{-1}$ follows easily by the symmetry condition. From Theorem 5, the $C(N, d)$ proper dual cones W_j^* cover almost every point in E^d exactly $\binom{N-1}{d-1}$ times. Therefore, $\sum_{j=1}^{C(N,d)} \mu(W_j^*) = \binom{N-1}{d-1}$, and the second half of the theorem follows by symmetry.

4. Remarks. The total number of non-empty cones W, proper dual cones W^*, and k-faces of these cones have been determined and shown to be independent, up to general position, of the configuration of $x_1, x_2, \cdots, x_N$. Among the rays generated by each of the $x_1, x_2, \cdots, x_N$, the extremal ones are the 1-faces of W^*, the expected number of which appears in (3.3) and (3.4). Thus,

$$ER_1(W^*) = 2NC(N-1, d-1)/C(N, d) \tag{4.1}$$

and

$$\lim_{N\to\infty} ER_1(W^*) = 2(d-1). \tag{4.2}$$

As a special case, suppose N points are chosen at random on the surface of the unit sphere in E^3. Then, given that they all lie in some single hemisphere, the expected number of extreme points of their convex hull (taken with great circles on the surface of the sphere) does not grow without bound as N increases, but rather approaches the limit 4. This is perhaps surprising, particularly since the number of vertices can in no case be less than 3. For a comparison with the case of random points in the plane, where the expected number of extreme points goes to infinity, see [4] and [2]. On the other hand, the great circles having the N chosen points as poles partition the surface of the sphere into regions having an expected number of sides 4 as N goes to infinity. This agrees with the known result for regions formed by random lines in the plane [3].

Closer inspection of (4.1) reveals that the expected number of extreme vectors of a random proper cone generated by N random vectors in E^d monotonically increases to $2(d-1)$ as N increases to infinity. We also remark that the asymptotic expected number of k-faces of W, given in (3.2), corresponds to the number of $(k-1)$-faces of a $(d-1)$-cube [6]. Loosely speaking, the "expected" cross section of W is a $(d-1)$-cube.

[4] Here μ has no connection with the probability mechanism generating the vectors X_i, and in particular does not have to satisfy any symmetry conditions.

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