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GENTLE DIVERSIONS

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In and Out of the Dazzling Continued Fraction

Communications engineers accumulate a whole arsenal of tools in their travels, extending from such powerful methods as matrix algebra and the Fourier transform, to handy rules of thumb for bandwidth, insertion loss, and antenna gain. Some tools don't get the attention they deserve, and the subject of this month's column is a prime example. It's not that continued fractions have been ignored; the next paragraph gives a wealth of published examples of their use in engineering and mathematics. But they are simple and fascinating, and deserve to be examined as entities in their own right, rather than plucked out of the hat as the need arises. They also lend themselves to pleasing pastimes and puzzles, as I hope you'll shortly agree.

SOME APPLICATIONS OF CONTINUED FRACTIONS

Here I mention, with sample references, some of the numerous and fruitful applications continued fractions (CF's) have found in our general field of interest. They are perhaps best known to electrical engineers through classical filter synthesis techniques leading to ladder networks [1] (including synthesis of irrational-valued impedances [2]), and more recently have been used to synthesize both one- and two-dimensional digital filters [3],[4]. Matrix continued fractions have appeared in control theory [5],[6], yielding generalized feedback structures and system models with reduced complexity, and in the same context have been used in the inversion of matrices of polynomials [7]. They have also been useful tools in the solution of difference equations [8],[9], systems of linear equations [10], and in the solution of such differential equations as Schrödinger's [11] and the (scalar) Riccati equation [8],[12]. Stone [13] has used them to develop efficient algorithms for solving equations involving tridiagonal matrices, such as appear in spline interpolation work. In coding theory, continued fractions have been used in connection with shift-register sequences and decoding Goppa codes [14], and they play a prominent role in the version of Euclid's algorithm which Berlekamp [15] uses in his remarkable algorithm for decoding BCH codes. The theory of CF's also touches base in a marvelous way with ergodic theory [16]. In the computer area they are used to evaluate special functions [9],[17],[27],[30],[33], to study Padé approximants [9], in the well-known "quotient-difference" algorithm [18], and to solve polynomial equations [8], [9]. Algorithms have been studied for the efficient manipulation of CF's themselves [19], and the whole issue of hardware design based on their use has been wrestled with [17]. The list, even in very current literature, seems endless, and is by no means complete here. Please send in others as you come upon them, so I can share them in future columns. Our main interest here will be with pleasing aspects of their basic nature and their very first historical applications, that is, to number theory and to approximation of irrational numbers.

To go beyond the brief introduction here, you'll find Olds [20], Dantzig [21], Chrystal [32], and Khovanskii [8] to be excellent references, and of course, Knuth [22] always does a splendid job. Hardy and Wright [23] is superb and eloquent.

SIMPLE CONTINUED FRACTIONS

A continued fraction expansion basically provides an alternate representation for a number, which often reveals more of the number's nature than does its decimal or binary representation. It's easy to expand a number like 14/11 in a so-called "simple" continued fraction. Peel off the integral part: $14/11 = 1 + 3/11$, and then write the remainder as $1/(11/3)$. Peel off the integral part of 11/3, i.e., $3 + 2/3$, and operate on 2/3 in the same fashion. Invert and peel until the process terminates: it always will for a rational number, the ratio of two integers. Now put it all together as $14/11 = 1 + 1/(3+1/(1+1/2))$, or

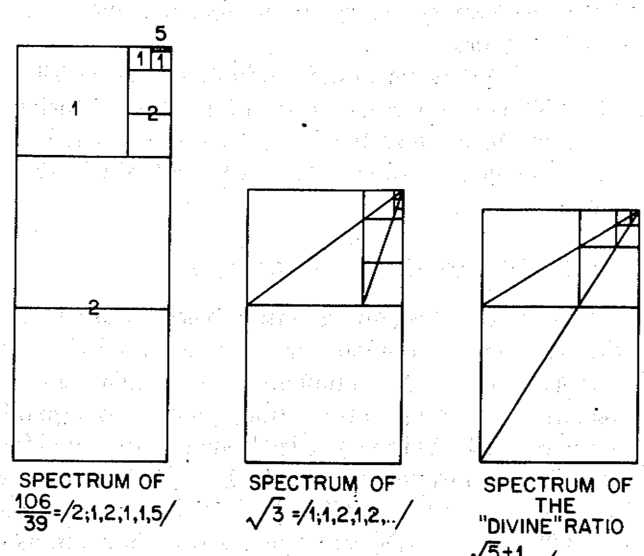
$$14/11 = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{2}}} \quad . \qquad (1)$$

To save precious space, write this as $14/11 = /1;3,1,2/$. More generally, for the simple continued fraction of any number w, we write

$$w = /\, a_0;\ a_1, a_2, a_3, \cdots / \qquad (2)$$

meaning $w = a_0 + 1/(a_1 + 1/(a_2 + 1/(a_2 + \cdots)))$. The semicolon sets off the integral part a_0, while the remaining a_k's, the so-called "partial quotients," are positive integers. In general, every rational number has a terminating simple CF. Its CF is also unique except that a final 1 may be incorporated by inspection into the preceding partial quotient, e.g., /3;2,5,1/=/3;2,6/. Dantzig [21] calls $a_0, a_1, \cdots$ the "spectrum" of w, a handy term I'll use here too. For practice, check these: a) 113/355 = /0;3,7,16/; b) $-2/7$ = /−1;1,2,2,/; and c) perhaps using the invert-and-peel algorithm on your pocket calculator—until its limited precision clobbers your efforts, $\sqrt{2}$ = /1;2,2,2,2,2,2,$\cdots$/. $\sqrt{2}$ has an infinite spectrum, but it's easily checked as we'll see below. Also, to warm up, try finding the spectra of both $2w$ and $1/w$ in terms of the spectrum of w in (2).

A nice geometric view [21] of the spectrum appears in Fig. 1. Begin with a rectangle of unit height and length w. Delete as many squares as possible from one end. From the rectangle that remains do the same: delete as many squares as possible. Continue this process until nothing remains (or until your pencil is too pudgy). Now the number of squares in any one of the successive rectangles gives the corresponding term in the spectrum of w. *Puzzle #1:* Why does it work, and what can be said about the general nature of those diagonals? (See a recent issue [26] of this column for more on the marvelous Golden Section.)



SPECTRUM OF $\frac{106}{39}$=/2;1,2,1,1,5/

SPECTRUM OF $\sqrt{3}$=/1;1,2,1,2,../

SPECTRUM OF THE "DIVINE" RATIO $\frac{\sqrt{5}+1}{2}$=/1;1,1,1,.../

Fig. 1. Geometric view of a number's spectrum.

If you have access to the programming language APL, and enjoy its glittering elegance, you may wish to evaluate continued fractions from their spectra according to the expression [31]:

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⍎2↓,'+','÷',⍕A∘.+,0
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where vector A contains the spectrum of w. It is also interesting to write an algorithm to go the other way, to produce the spectrum of a number from its value. Please send in any particular elegant solutions.

Every irrational number has a unique nonterminating CF, and vice versa: the CF $/a_0;a_1,a_2,\cdots,a_N/$ always converges to an irrational number as N goes to infinity, for any choice $a_0, a_1, a_2, \cdots$ of positive integers, i.e. it never blows up or oscillates [23]. The spectra of some irrationals exhibit obvious patterns. The Golden ratio $\emptyset = 1/2(1+\sqrt{5})$ = /1;1,1,1,$\cdots$/ in Fig. 1 is the simplest of all, while [22] $e = 2.718281828 \cdots$ = /2;1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,$\cdots$/, showing how much simpler a number's spectrum can be than its decimal expansion.

Other numbers have no apparent overall structure at all, e.g., [22]

$$
\begin{aligned}
2^{1/3} &= /1;3,1,5,1,1,4,1,1,8,1,14,1,10,2,\cdots/ \\
\pi &= /3;7,15,1,292,1,1,1,2,1,3,1,14,2,1,1,2,2,2,2,1,84,\cdots/.
\end{aligned}
\tag{3}
$$

(No less a light than John Von Neumann [24] was interested in the ability of a machine to evaluate the CF of $2^{1/3}$, while recently [25] the CF of π has been computed to 19 945 places. You'll be happy to learn that there are ten partial quotients larger than 2000 in there, the largest being 20 776. The smallest missing partial quotient is 103.) The famous Euler constant [22] $\gamma = 0.57721566 \cdots$ = /0;1,1,2,1,2,1,4,3,$\cdots$/ appears in a multitude of number theoretic relations, yet it is still not known whether γ is rational or irrational. A recent computer study by Brent [29] found its first 20 000 partial quotients, and concluded that if γ is rational, then its denominator must be at least 10 000 digits long!

There is an interesting class of irrationals, the "quadratic surds," which have pleasing spectra. Quadratic surds are numbers of the form $(A \pm \sqrt{D})/B$ where A, B, and D are integers, and D is not a perfect square. (Equivalently, these surds are irrational solutions to quadratic equations having rational coefficients.) For these and only these creatures, the spectra repeats forever after some point, as in $(24-\sqrt{15})/17$ = /1;5,#2,3#/, where #2,3# means 2,3,2,3,2,3,$\cdots$. As an easy exercise find an expression for w = /b;#a,b#/, which yields CF's for many familiar numbers, including the Golden Section.

The special quadratic surds $\sqrt{N}$ where N is an integer but not a perfect square are even prettier: $\sqrt{N}$ = /a_0;#$a_1,a_2,a_3,\cdots,a_3,a_2,a_1,2a_0$#/. That is, the periodic part of their spectrum is symmetrical except for the last term $2a_0$, which is always twice the integral part, e.g., $\sqrt{29}$ = /5;#2,1,1,2,10#/, and $\sqrt{40}$ = /6;#3,12#/. There is even a class of numbers, the "reduced quadratic surds," which have a purely periodic spectrum [20].

MORE GENERALLY . . .

If one loosens up the form of CF's, and permits "numerators" other than unity and denominators other than integers, some recalcitrant numbers can be forced to reveal more of their nature. The notation is a bit more complicated, so we use another classical form [8]:

$$
x = a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +} \cdots \tag{4}
$$

with lowered plus signs to denote $x = a_0+b_1/(a_1+b_2/(a_2+b_2/(a_2+\cdots)))$. Such expansions are, of course, not unique: for instance, you can double b_1 in (4) and then doctor other terms so that the value of x does not change. How? Also, try to show the following result (for any a,b as long as a^2+b is positive):

$$\sqrt{a^2+b} = a + \frac{b}{2a+}\frac{b}{2a+}\frac{b}{2a} \quad + \cdots.$$

There is also

$$e = 2 + \frac{1}{1+}\frac{1}{2+}\frac{2}{3+}\frac{3}{4+}\frac{4}{5+}\frac{5}{6+}\cdots,$$

while π reluctantly permits [20] (due to Stern in 1833)

$$\pi/2 = 1 - \frac{1}{3-}\frac{2\cdot3}{1-}\frac{1\cdot2}{3-}\frac{4\cdot5}{1-}\frac{3\cdot4}{3-}\frac{6\cdot7}{1-}\frac{5\cdot6}{3-}$$

as well as [20] (Lord Brouncker, circa 1658)

$$4/\pi = 1 + \frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\frac{7^2}{2+}.$$

In the new Canadian periodical *Crux Mathematicorum*, vol. 2, p. 228, there appears a startling CF for 1:

$$1 = \frac{2}{1+}\frac{3}{2+}\frac{4}{3+}\frac{5}{4+}\frac{6}{5+}\cdots.$$

Note its deceptive similarity to the CF for e above. Also, draw a vertical line through any + sign in it, and the CF to the right also has value 1! (Why?)

If some ingredients are allowed to be variables, one can obtain some "gee whiz" representations for functions, which have been very seriously studied both in mathematical and computer circles [8],[27],[32]. Some of these expansions (suitably truncated) are actually used in commercial software today to evaluate such functions [30].

$$e^z = \frac{1}{1-}\frac{z}{1+}\frac{z}{2-}\frac{z}{3+}\frac{z}{2-}\frac{z}{5+}\cdots+\frac{z}{2-}\frac{z}{(2n+1)+}\cdots$$

$$\log(1+z) = \frac{z}{1+}\frac{1^2z}{2+}\frac{1^2z}{3+}\frac{2^2z}{4+}\frac{2^2z}{5+}\frac{3^2z}{6+}\frac{3^2z}{7+}\cdots$$

and there is the wonderful connection between the tangent and its hyperbolic cousin [20]:

$$\tan(z) = \frac{z}{1-}\frac{z^2}{3-}\frac{z^2}{5-}\frac{z^2}{7-}\cdots$$

$$\tanh(z) = \frac{z}{1+}\frac{z^2}{3+}\frac{z^2}{5+}\frac{z^2}{7+}\cdots.$$

The first was found by Lambert in 1770, and the second by Gauss in 1812. Communications engineers must frequently evaluate the normal probability integral: the area from x to infinity under $g(t)=\exp(-t^2/2)/\sqrt{2\pi}$. It has several CF's [32], one of which is

$$g(x)\left(\frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\frac{4}{x+}\cdots\right).$$

One can go even further: with suitable restrictions, the a_k's and b_k's of (4) can be matrices [5],[12], or indeed elements of any field: one must only insist that addition, multiplication, and inversion be well defined for the creatures you put in (4). But let's stick with simple continued fractions here given by (2) where the a_k's are positive integers: there's plenty to say about even these seemingly elementary forms.

CONVERGENTS

Much of the beauty surrounding continued fractions comes from their convergents. The nth convergent of a CF is formed by lopping off the spectrum after the nth term, or equivalently setting a_{n+1} to infinity. For $w = /a_0;a_1,a_2,\cdots/$ the nth convergent is $w_n = /a_0;a_1,a_2,\cdots,a_n/$. For example, $w_0 = a_0$ and $w_1 = (a_0a_1 + 1)/a_1$. Convergents are the keystone in any rigorous analysis of infinite CF's since limits are essential, but they also prove very useful in discussing how closely a CF approximates a given number of interest. For instance, one can compute by simple algebra the first few convergents for π in (3):

$$3, 22/7, 333/106, 355/113, 103993/33102, \cdots.$$

The approximation 22/7 is very famous, while 355/113 is remarkably close: accurate to six figures! The astounding mathematician Ramanujan discovered that 2143/22 is startlingly close to π^4. For practice, fiddle with convergents of π^4 and fourth powers of those for π.

It would be handy to have a way of calculating new convergents from previous ones without having to begin each time and build up the whole CF, as when you want to find /3;7,15,292/ from /3;7,15/. Happily, there is a simple recursive method that does the trick. It's attributed to Newton's teacher John Wallis, 1655, who also was the first to use the term "continued fraction." It's easily checked by induction. You can generate all the convergents by building the two integer sequences P_k and Q_k, $k=0, 1, 2,\cdots$, starting with $P_{-2}=0, P_{-1}=1, Q_{-2}=1, Q_{-1}=0$, and using

$$\begin{aligned} P_k &= a_k P_{k-1} + P_{k-2} \\ Q_k &= a_k Q_{k-1} + Q_{k-2} \end{aligned} \tag{5}$$

The kth convergent w_k is then just their ratio $w_k = P_k/Q_k$. [Check out the first few convergents of π above from its spectrum in (3).] As an example, if w is the Golden Section again, then all a_k's are 1, and both P_k and Q_k form Fibonacci sequences [26], so the convergents are $1/1, 2/1, 3/2, 5/3, 8/5, 13/8\cdots$. *Puzzle #2:* Show that the ratio P_k/P_{k-1} has CF: $/a_k;a_{k-1},a_{k-2},\cdots,a_0/$, the spectrum being backwards! The P_k's and Q_k's can each be viewed as outputs of a second-order digital filter: only the initial states are different for the sequences. One can also write the matrix form [28]

$$\begin{pmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.$$

The determinant $P_kQ_{k-1} - P_{k-1}Q_k$ of this matrix can be shown to be alternately $+1$ and -1 (due to Huygens around 1690, who used CF's to approximate the proper design for the toothed wheels of a planetarium), which immediately shows that P_k and Q_k are always relatively prime [23] (since any integer that divides both P_k and Q_k must also divide ± 1). Hence, every convergent $w_k = P_k/Q_k$ is automatically in lowest terms.

There is a pretty geometric view of convergents due to Felix Klein around 1897 [20], and illustrated in Fig. 2. Suppose a positive irrational number w as in (2) has convergents $w_k = P_k/Q_k$, $k=0,1,2,\cdots$. Plot the line $y = wx$ (the case where $w = \phi$ is used in Fig. 2) and put in pins at all integer pairs of coordinates as shown. Imagine a piece of thread tied at an infinitely remote point on line $y = wx$ and hold it taut at the origin. Now moving your hand toward the left, the thread will catch on certain pins; moving it to the right it catches on others. Here is the neat result: when you move to the left it catches on the odd convergents, with coordinates $(Q_1,P_1),(Q_3,P_3),\cdots$, and in going to the right it catches on the even convergents. This gives a clearer picture of how closely the convergents approximate w. *Puzzle #3:* Can you show that the line from (0,0) to (Q_k,P_k) is parallel to the line from (Q_{k-1},P_{k-1}) to (Q_{k+1},P_{k+1})?

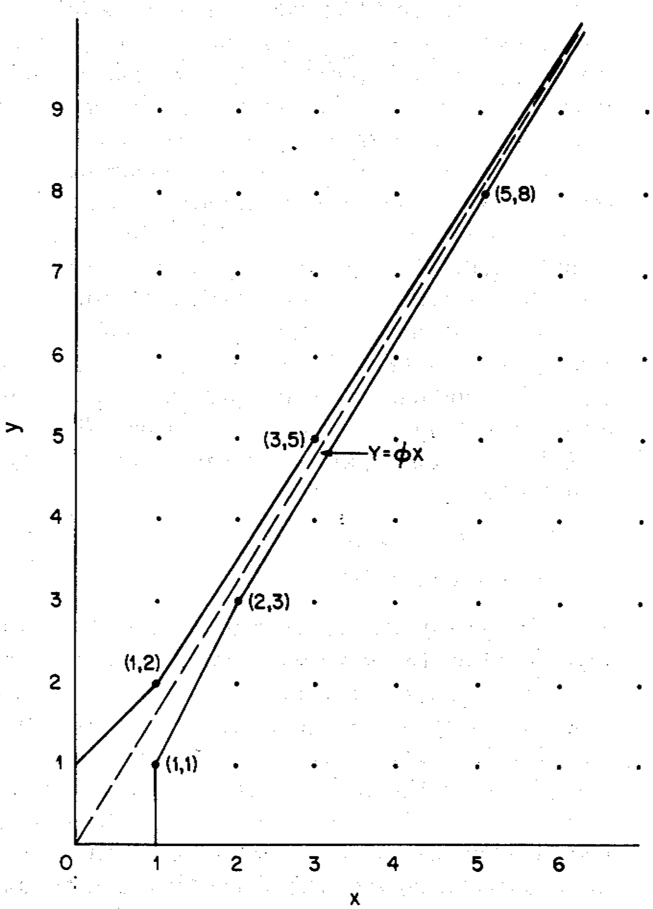

Fig. 2. Klein diagram for ϕ.

ON CALENDARS

There seems to be an intriguing connection between convergents and some calendar schemes, discovered by Dave Kelly, a delightful mathematician at nearby Hampshire College. It speaks to a possible basis upon which the Hebrew and Mohammedan calendars were constructed to cope with leap days. Astronomically, the solar year is 365.2422 days long, while the lunar month is 29.53059 days. The Mohammedan calendar is built around 12 lunar months, which is 354.36708 days, and the issue is how to sprinkle leap days so that after a number of years the months and years get synchronized again. Their solution is to have six months of 30 days alternating with six months of 29 days (making 354 days), and then to insert 11 leap days over each 30-year period. The question is, why choose 11 days in 30 years? Dave expanded 354.36708 in a CF and looked for the best approximation for a given sized denominator. It's the sixth convergent, 354 11/30! The Hebrew calendar, on the other hand, attempts to synchronize the solar with the lunar periods. The average number of lunar months in a solar year is 12.36827, which when expanded in a CF, has fifth convergent 12 7/19. Lo and behold, the Hebrew calendar postulates a cycle of 19 years, in seven of which a leap month is inserted! If you have other information and conjectures along these lines, please send them in.

Convergents are intimately connected to Diophantine equations (e.g., find all *integer* solutions to $13x + 17y = 300$), and to Euclid's algorithm, a much-used method for computing the greatest common divisor (gcd) of two integers (i.e., the largest integer that divides them both). Knuth [22] has studied the execution time required for Euclid's algorithm, and reports, among many other things, that it takes the longest time when the two integers are consecutive Fibonacci numbers! Berlekamp [15] developed a very efficient algorithm using convergents for applying Euclid's algorithm to polynomial equations over a finite field, with powerful applications to decoding BCH codes. He showed how to implement the algorithm for the case of binary arithmetic using simple gates and shift registers.

Suppose you must find the gcd of 6381 and 5163. Simply expand 6381/5163 into its CF, /1;4,4,5,2,1,1,3/, and calculate the P_k's and Q_k's using (5). The answer is based on the next to last ones: $P_6 = 592$ and $Q_6 = 479$, and is given by $(6381)(479) - (5163)(592) = 3$. The gcd is therefore 3. In general, the gcd of A and B, denoted (A,B), is $(A,B) = (-1)^n(AQ - BP)$ where n is the length of the spectrum of A/B and P/Q is its next to last convergent. Note that you get a bonus: it is known that the gcd of any pair of integers is some linear combination of the numbers: $(A,B) = cA + dB$ for some integers c,d. This method actually produces the values c,d, which is of great value in many coding situations [15].

Convergents are of considerable interest in approximating irrational numbers with "simple" rationals—ones with "small" denominators. Successive convergents w_k

of w "squeeze down" on the number w. In fact, the convergents w_k with an even index k are always less than w and form an increasing sequence, while those with k odd are larger than w and form a decreasing sequence. For the convergents of π above, these ideas are exhibited by the inequalities: $3 < 333/106 < 103993/33103 < \cdots < \pi < \cdots < 355/113 < 22/7$. The approximation error for the kth convergent can be shown to satisfy $|w - w_k| < 1/Q_k Q_{k+1}$, and since Q_k is an increasing integer sequence, w_k rapidly approaches w. This approach is more sluggish for the Golden Section than for any other number! (*Puzzle #4: Why?*) $\emptyset$ is beloved as the "worst" or "simplest" number of all in this area of mathematics.

There is a fascinating and extensive lore about approximating irrational numbers with rationals; here is just a glimmering of it. Fact: *any* irrational number has an *infinity* of rational approximations p/q that satisfy $|w - p/q| < 1/q^2\sqrt{5}$, i.e., there are infinitely many integer "denominators" q (with attendant numerators p) such that as you make q larger and larger, the approximation error promises to decrease as fast as $1/q^2\sqrt{5}$. But it turns out that $\sqrt{5}$ is the biggest constant you can use. If you try to replace it with a larger one, then there is at least one irrational (guess who is one of the culprits!) for which only a finite number of candidates p/q will work: as you make q bigger now, the approximation error just doesn't go down fast enough, and the process pinches off.

Enough said. There are many other interesting points I could aim you at, such as the fact that for "almost every" number x the geometric mean $(a_1 a_2 \cdots a_n)^{1/n}$ of its partial quotients approaches the same constant, and that this can be deduced by a surprising appeal to ergodic theory [16], but I won't. I'll close instead with some additional puzzles for your amusement, a project in musical tone generation, and solutions to puzzles in the September issue. Puzzle #9 was sent in by Tom Cover of Stanford, and may be more challenging. Please send in your solutions to some or all of the eight puzzles here within two weeks of receiving this issue, so I can report on who solved what, and in some cases, how. And, of course, send in your own puzzles, suggestions for topics, or even whole diversion articles!

Puzzle #5: Find the exact interval on the real axis whose points have a CF expansion of the form /2;1,2,1,1,4,1,1,$\cdots$/, in which only a_0 through a_7 are specified.

Puzzle #6: Solve in distinct positive integers: /0;#a,b#/ − 3/0;#c,d#/ = 1/2 (from *Pi Mu Epsilon*, 1978).

Puzzle #7: Give examples of irrational numbers less than one for which four decimal digits in the continued fraction representation give greater accuracy than four in the usual decimal expansion. Generalize.

Puzzle #8: (hard) Find an exact expression for the number having the CF: /0;1,2,3,4,5,$\cdots$/.

Puzzle #9: (by T. Cover) Let $X_1, X_2, \cdots$ be independent identically distributed random variables drawn according to a positive density function on the real line. Let $X^*_n = (1/n)(X_1 + X_2 + \cdots X_n)$ denote the sample mean for the first n samples. For *any* real number t, show: the probability that $|X^*_{2n} - t| < |X^*_n - t|$ always exceeds 1/2. Thus, the $2n$-sample mean is *strictly* stochastically closer to any point on the real line than is the n-sample mean. If we set $t = E(x)$, we seem to be saying that there really is a restoring force to the mean in the law of averages.

Project: Suppose you wish to synthesize the 12 notes of the equitempered musical scale which have frequencies well known to be related by twelfth roots of 2: $f_k = 2^{k/12} \cdot f_0$, $k = 0,1,2,\cdots,11$ for some initial tone at f_0. Your machine generates tones which unfortunately vary *linearly* with a control number x: $f = (1+x)f_0$, and you can store values of x only to 8-bit accuracy. How should you choose the 12 required 8-bit codewords to "best" approximate the 12 frequencies, and how can continued fractions be useful in this process? Would you be better off if your machine generated *periods*, the reciprocals of the frequencies required, linearly in x?

SOLUTIONS TO PUZZLES IN THE SEPTEMBER 1978 ISSUE

1) For the logarithmic spiral $r = d^{\theta}$, d must be $\emptyset^{(2/\pi)}$, and the points $H, I, J, K, \cdots$ are cut by another spiral: $r = (\emptyset^{3/2}\, 2^{-1/2})\emptyset^{2\theta/\pi}$.

2) Cover a unit circle with five smallest disks: look, in the figure shown there, at the triangle formed by the center of the unit disk, the center of one of the circles, and some intersection of these two circles. By symmetry, it's a golden triangle, so the required radius is $1/\emptyset$.

3) Let the radii of the two circles be in ratio a to b, and denote the distances from 0 to the centroids of the circles and lune by A, B, and C, respectively. Now the moments of these figures about 0 are ga^2A, gb^2B, and $g(a^2 - b^2)C$ for some constant g, and since moments add: $a^2A - b^2B = (a^2 - b^2)C$. Divide by b^2B and note $A/B = a/b$ and $C/B = 2$. Put it all together and solve to get $a/b = \emptyset$.

4) To express the sum of the first n Fibonacci numbers in terms of F_{n+2}, use the basic definition and write, $F_{n+2} = F_{n+1} + F_n$. Do the same to F_{n+1}, writing it as $F_n + F_{n-1}$. Keep doing it, until you arrive at $F_{n+2} = F_n + F_{n-1} + \cdots + F_1 + F_2$, so the required sum is just $F_{n+2} - 1$.

5) Factor n^n from all three terms to get $1 + (1 + A/n)^n = (1 + 2A/n)^n$, which approaches $1 + e^A = e^{2A}$, and recognize the solution $e^A = \emptyset$.

6) View the input driving sequence F_n of the digital filter as the output of another filter having transfer function $1/(1 - z^{-1} - z^{-2})$, a Fibonacci generator. Now just cascade the two filters and find the overall impulse response.

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Base drive—primitive motivation
Transformer—successful evangelist
Pulse delay—cardiac arrest
Johnson counter—Howard's cafeteria
Current mode logic—contemporary fad in mathematics
Phase lock—frozen expression
Delay line—debtor's story
Semiconductor—part-time railwayman
Disk drive—slap shot
Subroutine—undersea maneuvers
Star-connected—well known in Hollywood
Baud rates—hooker's fee schedule
CMOS—underwater vegetation
Meter movement—search for parking spot
Bonding wire—cheap wedding ring
Battery charger—prosecutor in assault case
Slip ring—illegal negligee pusher
Standard cell—ordinary prison accommodation
Down counter—unemployed accountant
Vidicon—TV commercial
Minimum-loss pad—low-rental apartment
Data extractor—census taker
Banana plug—fruit marketing board ad
Crimp termination—strangulation
Truth table—torture rack
Threshold logic—door-to-door salesman's argument
Majority logic—democratic process

Pas also sent the following short article, by Prof. Robert L. Brown in the Department of Statistics at the University of Waterloo. It first appeared in *QUEST*, May 1979, and is reprinted with permission. If you feel the urge to write a short development of the underlying probability theory, please send it in to me.

HOW TO MAKE THE BEST CHOICE

Robert L. Brown

You're driving down the highway in a strange province and it's time for lunch. You want to choose the best possible restaurant and yet you realize that once you've passed any candidate, you will not be willing to drive back to it. How can you maximize your chances of choosing the best?

There's a simple way to increase them—dramatically. It is based on a complex mathematical theory, but in practice, it's easy. Let's say you'll consider the next eight restaurants you come to. You look at the first three but you don't stop. You then stop at the next restaurant that looks better than the best of the first three. Chances are, you'll have found the best restaurant of all eight you were prepared to consider.

This method is based on an interesting theory whose solution has far wider applications than is obvious at first glance. Many academics have written papers on it and the earliest formulas and most refined results were produced by Dr. H. Robbins and Dr. Y.S. Chow of Columbia University.

Their solution, in its most general form, is as follows: If you are faced with trying to make the best choice out of "n" random possibilities, you should consider "s" candidates and then choose the first subsequent possibility that is better than the best candidate. The value of s can be found by dividing n by $e = 2.7183$. Having done so, the probability will never be less than 0.368 that you will make the best choice.

Let's go through the restaurant example with numbers. You do not want to look at any more than eight restaurants. The quality of the restaurants is random. The theory says that you should look at $8/e$ or three restaurants as candidates, remembering the best of these three, but making no attempt to stop at any of them. Then, the first restaurant after these that is better than the best of your three candidates is where you should stop. The probability is greater than $1/e$, or 36.8 percent, that you will have chosen the best restaurant. (In fact, with only eight choices, the probability is better than 40 percent.)

An even simpler example may convince you that the theory has merit. Consider a situation where there are only three possible restaurants: A (the best), B (good), and C (the worst). They can be arranged in six ways, which are equally likely. The theory says you should look at one candidate, but not stop, and then choose the first restaurant after that that is better than the candidate. The system will work as shown below.

Order of Restaurants	*Theory's Selection*
C—B—A	B
C—A—B	A
B—A—C	A
B—C—A	A
A—C—B	B
A—B—C	C

The last order forces you to eat at restaurant C (the worst) because you cannot go back. Note that the theory produces the best choice three times out of six and the worst choice only once out of six. A random selection will find the best restaurant one time in three, but also will find the worst restaurant one time in three. Other values of n and s, with their resultant probabilities, are found in the following table:

n Total Number of Choices	s Number of Candidates	Probability of Making the Best Choice
3	1	50 percent
4	1	45.8 percent
5	2	43.3 percent
10	3	39.9 percent
20	7	38.4 percent
50	18	37.4 percent
100	37	37.1 percent
∞	n/e	$1/e \cong 36.8$ percent

You can find other examples of applications for the theory that are both humorous and, with any luck, profitable.

A bachelor girl decides it's time to get married. She figures that a maximum of 10 men will propose to her but she realizes that once she has rejected a proposal, the man will probably not try again. What strategy should she follow to maximize her chances of accepting the best man, and what is the probability that she will succeed?

The theory is that the girl should reject the first three proposals of marriage, remembering which of these three men was her favorite. She should then marry the first man who is more favorable than the best of the three candidates. The probability that she will thus marry the best man of the 10 is 40 percent, or two chances in five. Not bad.

Finally, assume you are about to invest in the stock market. It is your belief (and many highly educated economists would agree) that the stock market behaves in a purely random fashion. You want to maximize the profit on your investment but you must have your money out before five years have gone by. What is your best strategy?

In this situation, if the stock trades actively at all, it is perfectly all right to apply the theory to a time variable. According to the theory, you should follow the stock for $5/e$ years, or one year and 306 days (one year and 10 months). At no time in that period should you make any attempt to sell the stock but you should record its highest selling price in that time. The first time after that that the stock can be sold at a higher price, you should sell. The probability is 36.8 percent that you will have sold at its highest point for the entire five-year period. The probability is not 43.3 percent (corresponding to $n = 5$), since the stock will have traded many times in that period (thus you use $n = \infty$ to determine the probability). Could you do as well?

It's an interesting and worthwhile theory. It teaches us that it is essential to wait and see, but it also shows us that a surprisingly small amount of waiting is necessary. Give it a try next time you're searching for a place to eat, investing in the stock market, or looking for a spouse. □

W. R. Bennett, very well known to us for his many important contributions to our field, has shed more light on the involvement of ellipses in astronomy, discussed here in the July 1979 column, and by Bill Pritchard in the January 1980 issue. He points out that (1) in the latter installment should read $M = E - e \sin E$, relating the "mean anomaly" M of a planet to the eccentricity of its orbit e, and its eccentric anomaly E. He also mentions that the true anomaly θ is related to E by $\tan(\theta/2) = \sqrt{(1+e)/(1-e)}\tan(E/2)$. Then he goes on with some fascinating insight and history:

"Second, contrary to the author's statement, Kepler's equation *is* explicitly solvable for both E and θ in terms of M. Lagrange gave the solution for E in 1770 in terms of what might be called a pre-Fourier Fourier series in M, which in modern notation is

$$E = M + 2 \sum_{n=1}^{\infty} \frac{J_n(ne)}{n} \sin nM.$$

In 1824, Bessel reformulated Lagrange's work to obtain explicit expressions for both E and θ, and was rewarded by having the coefficients in the Fourier series named after him. The series may also be regarded as a Kapteyn series in e. A complete treatment can be found in Watson, *Theory of Bessel Functions*, Cambridge, England, 1922, pp. 6, 13, 551-556.

Kepler's equation turns up from time to time in the analysis of various devices and methods used in communication systems. One example is the light valve, invented by E. C. Wente and described by D. MacKenzie in the *Bell System Technical Journal*, vol. 8, pp. 173-183, 1929. A moving film is exposed to light transmitted through a variable-width aperture formed by ribbons vibrating under control of the signal wave. Calculation of the density of exposure versus distance along the film when the ribbon motion is sinusoidal leads to Kepler's equation. The coefficients in the explicit solution give the amplitudes of the fundamental and harmonic components in the recorded response.

The equation is also encountered in the analysis of variable-velocity modulation of a cathode-ray beam as was proposed in the early days as an alternate method of constructing a television image. Another related case is that of pulse position modulation with natural sampling."

Finally, here is a bevy of problems ranging from very easy to rather challenging: The first four are taken from C. W. Trigg, *Mathematical Quickies*, New York: McGraw-Hill, 1967.

1) The axes of symmetry of two 1-in right circular cylinders (broom handles) intersect at right angles. What volume do the cylinders have in common? (It has been said that the formidable Charles P. Steinmetz figured this one out in his head in the time it took him to take his cigar out of his mouth.)

2) A die bearing the numbers 0, 1, 2, 3, 4, and 5 on its faces is repeatedly thrown until the total of the throws first exceeds 12. What is the most likely total that will be thus obtained?

3) Every person on earth has shaken a certain number of hands. Prove that the number of persons who have shaken an odd number of hands is even.

4) Find all of the relative maximum and minimum values of $(x^2 - 2x + 2)/(2x - 2)$ without using calculus.

5) Lew Franks came across this pretty graphical solution for a quadratic equation, due to Thomas Carlyle (1795-1881). To find the roots of $x^2 - gx + h = 0$, draw the circle with diameter terminating at (0,1) and (g,h) (Fig. 1). The roots x_1 and x_2 then lie on the intersection of the circle with the horizontal axis. Why does it work? Can you generalize to complex roots?

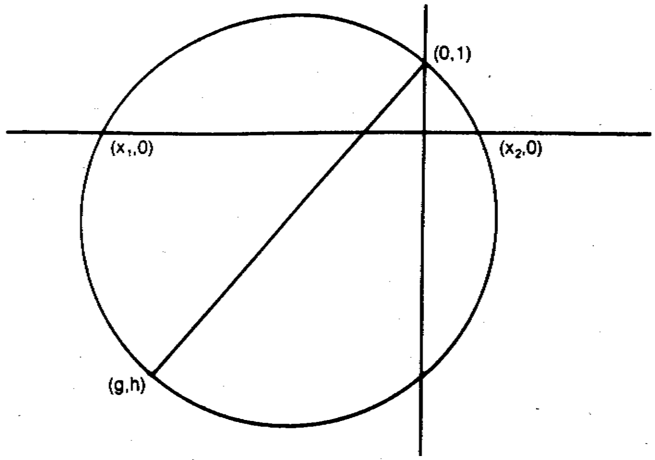

Fig. 1.

6) The number 3025 has the pleasant property that $3025 = (30 + 25)^2$. Find all other 4 digit numbers—having no repeating digits—with this property (sent in by Bob Maurer).

And Tom Cover, whose earlier puzzle on the restoring force of the law of averages (see this column, January 1979) has stimulated more responses than any other, sent in these 3 for you lovers of probability.

7) "Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that $H(X_0|X_{-1},X_{-2},\cdots,X_{-n})$ is equal to $H(X_0|X_1,X_2,\cdots,X_n)$. That is, the present has a conditional entropy given the past equal to the conditional entropy given the future.

This is true even though it is quite easy to concoct stationary random processes for which the flow into the future looks quite different from the flow into the past. That is to say, one can determine the direction of time by looking at a sample function of the process. Nonetheless, the conditional uncertainty of the next symbol in the future is equal to the conditional uncertainty of the next symbol in the past."

8) A random n-dimensional rectangular parallelipiped $[0,X_1] \times [0,X_2] \times \cdots \times [0,X_n]$ is formed, where $X_1,X_2,\cdots$ are i.i.d. RV's drawn according to a uniform distribution on the unit inteval [0,1]. Let L_n denote the length of the side of an n-cube with the same volume as the random parallelipiped.

Show that L_n converges with probability 1 to a constant c and find c.

9) Let $X_1,X_2,\cdots$ be Bernoulli with parameter p. Consider the two-hypothesis testing problem

$$H_1: p = \alpha$$

versus

$$H_2: p = 1 - \alpha:$$

Let both hypotheses be equally probable. Try to find a clever argument that $2n$ observations result in the same (Bayes) probability of error as do $(2n - 1)$ observations.