NOTES

THE PROBABILITY THAT A RANDOM GAME IS UNFAIR[1]

By Thomas M. Cover

Stanford University

1. Introduction and summary. To every matrix A of real valued payoffs there corresponds a unique quantity $V(A)$, the value of a game with payoff matrix A, defined by

$$V(A) = \min_y \max_x x^tAy = \max_x \min_y x^tAy,$$

where the maximum and minimum are taken over all x and y having nonnegative components summing to one. We wish to show that, for a relatively wide class of probability distributions on the class of all $m \times n$ game matrices, the probability $P(m, n)$ that the value of a random game is greater than zero is given by the cumulative binomial distribution

$$P(m, n) = (\tfrac{1}{2})^{m+n-1} \sum_{k=0}^{m-1} \binom{m+n-1}{k}.$$

The observation that, for any $\epsilon > 0$,

$$\lim_{m\to\infty} P(m, (1 + \epsilon)m) = 0,$$

$$P(m, m) = \tfrac{1}{2}, \qquad \text{for all m,}$$

$$\lim_{m\to\infty} P(m, (1 - \epsilon)m) = 1,$$

suggests that large rectangular games tend to be strongly biased in favor of the player having the greater number of alternatives. Thus we characterize the extent to which the "nonsquareness" of a game is reflected in a bias for one of the players. Our method involves an application of a theorem in combinatorial geometry due to Schläfli [5]. A discussion of the consequences of Schläfli's theorem in geometrical probability is given in [2], [3], [9].

2. Random work on random games. Goldman [4] has shown that "large matrices rarely have saddlepoints." More precisely, he shows that an $m \times n$ matrix of independent identically distributed random variables with a common continuous distribution function has a saddlepoint (an element simultaneously the minimum of its row and the maximum of its column) with probability $m!n!/(m + n - 1)!$. A ready generalization of this result is made via the Shapley-Snow theorem [6] which states that corresponding to a basic optimal strategy for either player in any matrix game there is a square submatrix (the kernel) whose associated game has the same value as the given game and which

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has basic optimal strategies that are optimal in the original game when augmented by zeros in the restored coordinates. Thrall and Falk [7] establish a geometric argument for the kernel as a generalized saddlepoint and investigate the probabilities of $k \times k$ kernels for games whose elements are i.i.d. uniformly on [0, 1], without, however, making a systematic evaluation of the resulting integrals.

The aforementioned results are probabilistic statements about the structure of random games. In special cases, statements about the values of random games can be developed. Thomas ([7], p. 88) and Thrall and Falk ([8], p. 366), for example, have observed that the value of a game of i.i.d. elements, conditioned on its having a saddlepoint, has the distribution of the nth largest of $m + n - 1$ i.i.d. random variables drawn according to the common distribution governing the elements of A. Thrall and Falk obtain partial results on determining the distribution of the value of random games the elements of which are i.i.d. random variables drawn according to the uniform distribution on [0, 1]. Thomas goes on to determine the distribution of the value of the game under these conditions for 2×2 games and, asymptotically as $n \to \infty$, for $2 \times n$ games. Thomas is also concerned with the values of n-stage random games. This aspect of his work is somewhat related to the work of Chernoff and Teicher [1] in which they determine the class of limiting distributions for the normalized minimax (or maximin) of independent identically distributed random variables. We find that even in the simplest cases, determining the distribution of the value of a random game is largely an unsolved problem.

3. Propositions. We shall determine $P(m, n)$ by first establishing a combinatorial proposition counting the number of game matrices with positive values derivable from A by negations of subsets of rows and columns. Let α (and β) range over $m \times m$ (and $n \times n$) diagonal matrices with diagonal elements ± 1. Thus the set of all matrices derivable from A by negations of subsets of rows and columns is the set of 2^{m+n} $\alpha A \beta$'s (of which at most 2^{m+n-1} are distinct).

PROPOSITION 1. *Let A be an $m \times n$ matrix every square submatrix of which is nonsingular. The number of games $\alpha A \beta$ for which $V(\alpha A \beta) > 0$ is $2 \sum_{k=0}^{m-1} \binom{m+n-1}{k}$. Since, by the Snow-Shapley theorem, $V(\alpha A \beta) \neq 0$, the remaining $2 \sum_{k=m}^{m+n-1} \binom{m+n-1}{k}$ games have $V(\alpha A \beta) < 0$.*

PROOF. We utilize a combinatorial geometric statement due to Schläfli [5]. Let $z_1, z_2, \cdots, z_k \in E^m$ be in general position (i.e., every m-element subset of $z_1, z_2, \cdots, z_k$ is linearly independent). Then Schläfli has proved that the set of inequalities $\delta_i w^t z_i > 0$, $i = 1, 2, \cdots, k$, is consistent for precisely

$$C(k, m) = 2 \sum_{j=0}^{m-1} \binom{k-1}{j}$$

of the 2^k sequences $\{\delta_1 = \pm 1, \delta_2 = \pm 1, \cdots, \delta_k = \pm 1\}$. Alternatively, precisely $C(k, m)$ polyhedral cones of the form $\bigcap_{i=1}^{k} \{w : \delta_i w^t z_i > 0\}$ are nonempty.

By the definition of the value of the game, it is clear that $V(\alpha A \beta) > 0$ iff there exists $x > 0$ such that $x^t \alpha A \beta > 0$, where all vector inequalities are meant to hold component-wise. Let $c_1, c_2, \cdots c_n \varepsilon E^m$ be the columns of A, and let $e_1, e_2, \cdots, e_m$ be the positive unit basis vectors of E^m. Consider polyhedral convex cones of the form

$$W(\alpha, \beta) = \bigcap_{i=1}^{n} \{w: \beta_{ii} w^t c_i > 0\} \bigcap_{j=1}^{m} \{w: \alpha_{jj} w^t e_j > 0\}.$$

Now $W(\alpha, \beta)$ is nonempty if and only if $V(\alpha A \beta) > 0$. For, if $x = \alpha w$, with $w \varepsilon W(\alpha, \beta)$, then $x > 0$ and $x^t \alpha A \beta = w^t A \beta > 0$, which implies that $V(\alpha A \beta) > 0$. On the other hand $V(\alpha A \beta) > 0$ implies there exists $x_0 > 0$ such that $x_0^t \alpha A \beta > 0$; hence $\alpha x_0 \varepsilon W(\alpha, \beta)$ and $W(\alpha, \beta) \neq \emptyset$. Thus the number of nonempty $W(\alpha, \beta)$'s is equal to the number of games $\alpha A \beta$ having positive values, and the set of strategies achieving positive values in the game $\alpha A \beta$ is the cone $\alpha W(\alpha, \beta)$.

Every m-element subset $\{c_{i_1}, c_{i_2}, \cdots, c_{i_k}, e_{i_{k+1}}, e_{i_m}\}$ of the set $\{c_1, c_2, \cdots, c_n, e_1, \cdots, e_m\}$ is linearly independent, as can be seen by expanding the determinant of the subset by minors with respect to the columns $e_{i_{k+1}}, \cdots, e_{i_m}$ and observing that the determinant of the resulting $k \times k$ submatrix of A is nonzero by assumption. Hence, the set $\{c_1, c_2, \cdots, c_n, e_1, e_2, \cdots, e_m\}$ is in general position in E^m, and by Schläfli's theorem precisely $C(m + n, m)$ of the cones $W(\alpha, \beta)$ are nonempty. This completes the proof.

Proposition 2. *Let A be an $m \times n$ matrix such that every square submatrix is nonsingular. Then the number of games $\alpha A \beta$ for which $V((\alpha A \beta)^t \alpha A \beta) > 0$ is $2^{m+1} \sum_{k=0}^{m-1} \binom{n-1}{k}$, and, since $V(A^t A) \geqq 0$, the rest have $V = 0$.*

Proof. The nonnegativity of V is established by matching strategy x to y:

$$V((\alpha A \beta)^t \alpha A \beta) = \max_x \min_y x^t \beta^t A^t A \beta y \geqq \min_y (A \beta y)^t A \beta y \geqq 0$$

Equality holds if and only if there exists a probability vector $y > 0$ such that $A \beta y = 0$. By the duality theorem of linear programming such a y exists if and only if there exists no x in E^m such that $x^t A \beta > 0$. By Schläfli's theorem, this system of n simultaneous inequalities in m unknowns is consistent for precisely $C(n, m)$ choices of β. Since this is true for each of the 2^m choices of α, the proof is complete.

4. Implications for random games. It is easy to generate probabilistic statements of Propositions 1 and 2. Let A be a random matrix such that every square submatrix is nonsingular with probability one. Then the relative frequencies implicit in the propositions become probabilities for any distribution on $\{A\}$ for which the conditional distributions of A, given the orbits $\{\alpha A \beta \mid \text{all } (\alpha, \beta)\}$, are uniform. (The orbits $\{A \beta \mid \text{all } \beta\}$ suffice in the case of Proposition 2.) Under these conditions on the distribution of A we have the following statements:

Proposition 3.
$$\Pr(V(A) > 0) = P(m, n),$$
$$\Pr(V(A) = 0) = 0,$$
$$\Pr(V(A) < 0) = P(n, m).$$

PROPOSITION 4. $\Pr(V(A^tA) > 0) = P(m, n - m)$,

$\Pr(V(A^tA) = 0) = P(n - m, m)$.

PROPOSITION 5. $\Pr(V(AA^t) > 0) = P(n, m - n)$,

$\Pr(V(AA^t) = 0) = P(m - n, n)$.

The above sufficient conditions on the distribution of A are satisfied, for example, in the special case where the elements of A are independent, symmetric, real valued random variables with continuous distribution functions. In particular, let the elements of the $m \times n$ matrix A be i.i.d. uniformly on [0, 1], as in the considerations of Thomas, Thrall, and Falk. Then Proposition 3, referred to the matrix A centered at its mean, yields $\Pr(V(A) > \frac{1}{2}) = P(m, n)$; $\Pr(V(A) = \frac{1}{2}) = 0$; $\Pr(V(A) < \frac{1}{2}) = 1 - P(m, n) = P(n, m)$.

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