

Max-Min Optimal Investing*

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Abstract

We solve the problem of tracking the best constant rebalanced portfolio computed in hindsight in a max-min optimal sense and relate our results to the pricing of a new derivative security which might be called the hindsight allocation option. This option pays the return of one dollar invested in the best constant rebalanced portfolio computed in hindsight.

1 Introduction

In practice one would expect the wealth achieved by the best constant rebalanced portfolio computed in hindsight to grow exponentially with a rate determined by market volatility. Even if the prices of individual assets are going nowhere in the long run, short term fluctuations in conjunction with constant rebalancing may lead to substantial profits. The intuition that the best constant rebalanced portfolio is a worthy performance target is motivated by the well known fact that if market returns are assumed independent and identically distributed from one day to the next, the expected utility, for a large class of utility functions including the log, is maximized by a constant rebalanced portfolio. The question, then, is to what extent can a non-anticipating investment strategy track the performance of the best constant rebalanced portfolio determined in hindsight?

We address this question from a distribution-free, worst case perspective with no restrictions on asset price behavior. Asset prices can increase or decrease arbitrarily, even drop to zero. We assume *no* underlying randomness or probability distribution governing asset price

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changes and instead focus on individual sequences of price changes. The analysis is best expressed in terms of a contest between an investor and nature. After the investor has selected a non-anticipating investment strategy, nature, with full knowledge of the investor's strategy (and its dependence on the past), selects that sequence of asset price changes which minimizes the ratio of the wealth achieved by the investor to the wealth achieved by the best constant rebalanced portfolio computed in hindsight for the selected sequence. The investor, wishing to protect himself from this worst case, selects an investment strategy which maximizes the minimum ratio. A theorem appearing in [1] determines the max-min optimal strategy as well as the max-min value of the ratio of wealths. This result is summarized in Section 3.

We then apply the above max-min result to the pricing of a new derivative security which could be called the hindsight allocation option. The hindsight allocation option with duration n has a payoff at time n equal to S_n^* , the wealth earned by investing one dollar according to the best constant rebalanced portfolio (the best constant allocation of wealth) computed in hindsight for the observed market behavior. This option might, for example, interest investors who are uncertain about how to allocate their wealth among stocks and bonds. By purchasing a hindsight allocation option, an investor would achieve the performance of the best constant allocation of wealth determined with full knowledge of future market performance.

2 Notation and definitions

We represent the behavior of a market of m assets for n trading periods by a sequence of non-negative, non-zero¹ price relative vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}_+^m$. We use \mathbf{x}^n to denote such a sequence of vectors. The j^{th} component of the i^{th} vector denotes the ratio of closing to opening price of the j^{th} asset for the i^{th} trading period. Thus an investment in asset j on day i increases by a factor of x_{ij} . Investment in the market is specified by a portfolio vector $\mathbf{b} = (b_1, \dots, b_m)^t$ with non-negative entries summing to one or $\mathbf{b} \in \mathcal{B}$, where

$$\mathcal{B} = \{\mathbf{b} : \mathbf{b} \in \mathbb{R}_+^m, \sum_{j=1}^m b_j = 1\}. \quad (1)$$

The entries of \mathbf{b} denote the fractions of wealth invested in each of the m assets. Thus, investing according to portfolio \mathbf{b}_i on day i increases (or decreases) wealth by a factor of

$$\mathbf{b}_i^t \mathbf{x}_i = \sum_{j=1}^m b_{ij} x_{ij}. \quad (2)$$

A sequence of n investments according to portfolio choices $\mathbf{b}_1, \dots, \mathbf{b}_n$ increases wealth by a factor of

$$\prod_{i=1}^n \mathbf{b}_i^t \mathbf{x}_i. \quad (3)$$

¹At least one non-zero component.

A constant rebalanced portfolio investment strategy uses the same portfolio \mathbf{b} for each trading day. This results in a wealth increase of

$$S_n(\mathbf{x}^n, \mathbf{b}) = \prod_{i=1}^n \mathbf{b}^t \mathbf{x}_i. \quad (4)$$

For a sequence of price relatives \mathbf{x}^n it is possible to compute the best constant rebalanced portfolio \mathbf{b}^* given by

$$\mathbf{b}^* = \arg \max_{\mathbf{b} \in \mathcal{B}} S_n(\mathbf{x}^n, \mathbf{b}) \quad (5)$$

and achieving a wealth factor of

$$S_n^*(\mathbf{x}^n) = \max_{\mathbf{b} \in \mathcal{B}} S_n(\mathbf{x}^n, \mathbf{b}). \quad (6)$$

The best constant rebalanced portfolio can only be computed with complete knowledge of market behavior up through time n ; it is not a non-anticipating investment strategy.

Definition 1 *A non-anticipating investment strategy is a sequence of maps*

$$\mathbf{b}_i : \mathbb{R}_+^{m(i-1)} \rightarrow \mathcal{B}, \quad i = 1, 2, \dots \quad (7)$$

where

$$\mathbf{b}_i = \mathbf{b}_i(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}) \quad (8)$$

is the portfolio used on day i given past market outcomes $\mathbf{x}^{i-1} = \mathbf{x}_1, \dots, \mathbf{x}_{i-1}$.

3 Max-min analysis

Consider a contest between an investor, who selects a non-anticipating investment strategy $\hat{\mathbf{b}}_i(\cdot)$, and nature, who selects the market sequence $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. The investor's wealth against sequence \mathbf{x}^n is

$$\hat{S}_n(\mathbf{x}^n) = \prod_{i=1}^n \hat{\mathbf{b}}_i^t \mathbf{x}_i. \quad (9)$$

Nature, with full knowledge of the investor's strategy selects \mathbf{x}^n to minimize the ratio of $\hat{S}_n(\mathbf{x}^n)$, the investor's wealth, to $S_n^*(\mathbf{x}^n)$, the wealth obtained the best constant rebalanced portfolio computed with complete knowledge of \mathbf{x}^n . How well can the investor protect himself from the worst case in terms of maximizing this minimum ratio? The following theorem appearing in [1] answers this question. For simplicity's sake the theorem is stated for $m = 2$ assets. The theorem generalizes easily to $m > 2$.

Theorem 1 (Max-min) *For all n ,*

$$\max_{\hat{\mathbf{b}}} \min_{\mathbf{x}^n} \frac{\hat{S}_n(\mathbf{x}^n)}{S_n^*(\mathbf{x}^n)} = 1 / \left(\sum_{k=0}^n \binom{n}{k} \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k} \right). \quad (10)$$

Let

$$V_n = 1 / \left(\sum_{k=0}^n \binom{n}{k} \left(\frac{k}{n} \right)^k \left(\frac{n-k}{n} \right)^{n-k} \right). \quad (11)$$

It can be shown that $V_n \leq 2/\sqrt{n+1}$ and $V_n \geq 1/(2\sqrt{n+1})$ for all n . Thus V_n behaves essentially like $1/\sqrt{n}$, indicating that the investor can track the performance of the best constant rebalanced portfolio to within a factor of \sqrt{n} for all sequences of price relatives \mathbf{x}^n .

The theorem is proved by showing that for an investment strategy to be max-min optimal against any \mathbf{x}^n it suffices for it to be max-min optimal against sequences of price relative vectors in which each vector has exactly one non-zero component. The max-min investment strategy achieving the maximum in (10) is specified explicitly in [1].

4 The hindsight allocation option

We apply the above results to the pricing of the hindsight allocation option, a derivative security which pays the wealth earned by investing one dollar according to the best constant rebalanced portfolio computed in hindsight for the observed market behavior. Let

$$\bar{H}_n = \frac{1}{V_n}, \quad (12)$$

where V_n is given by (11). We argue [1] that the price of the hindsight allocation option should be no higher than \bar{H}_n . Since no restrictions are placed on nature's choice in the above analysis, this upper bound is valid for any market model. Asset prices can drop to zero overnight or can increase arbitrarily. Consequently \bar{H}_n may be too high a price to pay for this option in markets experiencing less volatility. To gain more insight into this issue we use classical derivative security pricing theory to obtain the price of the hindsight allocation option for two much studied models of market volatility, the binomial lattice and continuous time geometric Brownian motion models.

4.1 Binomial lattice

We analyze this model for the case of a risky stock and a riskless bond with parameters $r \geq 0$, $u > r > d$. The bond increases daily by $1 + r$ while the stock fluctuates up and down by $1 + u$ and $1 + d$. The no-arbitrage price H_n of the hindsight allocation option for this model is shown in [1] to be

$$H_n = \sum_{p_u < \frac{k}{n} < p_u \left(\frac{u+1}{r+1} \right)} \binom{n}{k} \left(\frac{k}{n} \right)^k \left(\frac{n-k}{n} \right)^{n-k} \quad (13)$$

$$+ \sum_{\frac{k}{n} \leq p_u} \binom{n}{k} p_u^k p_d^{n-k} \quad (14)$$

$$+ \sum_{\frac{k}{n} \geq p_u \frac{u+1}{r+1}} \binom{n}{k} \left(p_u \left(\frac{1+u}{1+r} \right) \right)^k \left(p_d \left(\frac{1+d}{1+r} \right) \right)^{n-k}, \quad (15)$$

where

$$p_u = \frac{r-d}{u-d}, \quad p_d = 1 - p_u. \quad (16)$$

From (11) and (12)

$$\bar{H}_n = \sum_{k=0}^n \binom{n}{k} \left(\frac{k}{n} \right)^k \left(\frac{n-k}{n} \right)^{n-k}. \quad (17)$$

The similarity between \bar{H}_n and the first component of H_n (13) is evident. In fact, if u , d , and r satisfy $p_u < 1/n$ and $p_u((u+1)/(r+1)) > (n-1)/n$ then $\bar{H}_n - H_n < 2$. Thus, for certain parameter choices the bound on the price of the option is nearly achieved even though one of the assets, the bond, has zero volatility. Of course, for this choice of parameters, the stock price fluctuates significantly.

4.2 Geometric Brownian motion

In this subsection we give the price of the hindsight allocation option for the classical continuous time Black-Scholes market model with one stock and one bond. The stock price X_t follows a geometric Brownian motion² and evolves according to the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t \quad (18)$$

where μ and σ are constant and B_t is a standard Brownian motion. The bond price β_t obeys

$$d\beta_t = \beta_t r dt \quad (19)$$

where r is constant and therefore

$$\beta_t = e^{rt} \beta_0. \quad (20)$$

Let $S_t(\mathbf{b})$ be the wealth obtained by investing one dollar at $t = 0$ in the constant rebalanced portfolio $\mathbf{b} = (b, 1-b)^t$, where b is the proportion of wealth in the stock. Then $S_t(\mathbf{b})$ satisfies the stochastic differential equation

$$\frac{dS_t(\mathbf{b})}{S_t(\mathbf{b})} = b \frac{dX_t}{X_t} + (1-b) \frac{d\beta_t}{\beta_t}, \quad (21)$$

²Here X_t denotes a price, *not* a price relative.

which can be solved to give

$$S_t(\mathbf{b}) = \exp\left(-\frac{b^2\sigma^2t}{2} + b\left(\log\frac{X_t}{X_0} + \frac{\sigma^2t}{2}\right) + (1-b)rt\right). \quad (22)$$

That this solves (21) can be verified directly using Ito's Lemma. Notice that for fixed σ^2, r the wealth $S_t(\mathbf{b})$ depends on the stock price path only through the final price X_t .

The best constant rebalanced portfolio in hindsight at time T is obtained by maximizing the exponent of (22) for $t = T$ under the constraint that $0 \leq b \leq 1$. This results in

$$b_T^* = \max\left(0, \min\left(1, \frac{1}{2} + \frac{(1/T)\log(X_T/X_0) - r}{\sigma^2}\right)\right). \quad (23)$$

From the martingale approach to options pricing, the no-arbitrage price at $t = 0$ of the hindsight allocation option with duration T is given by

$$H_{0,T} = \beta_0 E_Q \frac{S_T(\mathbf{b}_T^*)}{\beta_T} \quad (24)$$

$$= e^{-rT} E_Q S_T(\mathbf{b}_T^*) \quad (25)$$

where Q is the equivalent martingale measure or the unique (in this case) probability measure under which X_t/β_t is a martingale.

It is well known [2] that under the equivalent martingale measure Q the stock price X_t obeys

$$dX_t = rX_t dt + \sigma X_t dB_t. \quad (26)$$

This and Ito's Lemma imply that under the equivalent martingale measure the expression $\log X_T/X_0$ appearing in the exponent of (22) is normally distributed with mean $(r - (1/2)\sigma^2)T$ and variance $\sigma^2 T$. The expectation (25) is then easily evaluated. The solution reduces to the surprisingly simple form

$$H_{0,T} = 1 + \sqrt{\frac{\sigma^2 T}{2\pi}}. \quad (27)$$

The price is affinely increasing in the volatility σ and increases like the square root of the duration T . This dependence on duration matches the \sqrt{n} growth of the discrete time upper bound \bar{H}_n and the binomial lattice price H_n .

5 Conclusions

This solves the problem of tracking the best constant rebalanced portfolio computed in hindsight in a max-min optimal sense. It is possible to track the best constant rebalanced portfolio to within a polynomial factor uniformly for all sequences of asset price changes. For

the case of two assets the shortfall relative to the best constant rebalanced portfolio grows only like \sqrt{n} , which is asymptotically negligible.

The max-min solution leads to an absolute upper bound on the price of the hindsight allocation option valid for any market model. To gain more insight into this option and the bound, we derived the no-arbitrage price of the hindsight allocation option for the bond–stock binomial lattice and geometric Brownian motion market models. The bound and the no-arbitrage price all share the same square root dependence on the option duration.

Finally, the hedging strategies implied by the no-arbitrage pricing theory may constitute viable growth rate optimal investment strategies with better performance than the max-min optimal strategy under less volatile market conditions. On the other hand, the max-min strategy and the universal portfolios analyzed in [3, 4] may be better alternatives under highly volatile conditions or when not much is known about the market.

References

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