# Unified Duality between Channel Capacity and Rate Distortion with State Information

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*Abstract* — **We show that the duality between channel capacity and data compression is retained when state information is available to the sender, to the receiver, to both, or to neither. We also present a unified theory for different cases of channel capacity with state information, which extends existing results to arbitrary pairs of i.i.d. state information $(S_1, S_2)$ available at the sender and at the receiver, respectively. The general formula $C = \max_{p(u,x|s_1)}[I(U; S_2, Y) - I(U; S_1)]$ assumes the same form as the Wyner-Ziv rate distortion function with state information. The Wyner-Ziv formula also unifies four special cases of the rate distortion problem with the state information.**

## I. General Duality and Duality Relationships in Special Cases

It is known that when state information $S^n = (S_1, S_2, \cdots, S_n)$, where $S_i$ are i.i.d. $\sim p(s)$, is available non-causally to the sender ($C_{10}$), to the receiver ($C_{01}$), to both ($C_{11}$) and to neither ($C_{00}$), channel capacities are as follows:

$$
\begin{aligned}
C_{00} &= \max_{p(x)} I(X;Y) && (1)\\
C_{11} &= \max_{p(x|s)} I(X;Y|S) && (2)\\
C_{01} &= \max_{p(x)} I(X;Y|S) && (3)\\
C_{10} &= \max_{p(u,x|s)} [I(U;Y) - I(U;S)] && (4)
\end{aligned}
$$

It is also known that the rate distortion function when state information is available to the encoder ($R_{10}$), to the decoder ($R_{01}$), to both ($R_{11}$) and to neither ($R_{00}$) are

$$
\begin{aligned}
R_{00}(D) &= \min_{p(\hat{x}|x)} I(X;\hat{X}) && (5)\\
R_{11}(D) &= \min_{p(\hat{x}|x,s)} I(X;\hat{X}|S) && (6)\\
R_{01}(D) &= \min_{p(u|x)} \min_{f} [I(X;U) - I(S;U)] && (7)\\
R_{10}(D) &= \min_{p(\hat{x}|x)} I(X;\hat{X}) && (8)
\end{aligned}
$$

where $f : \mathcal{S} \times \mathcal{U} \to \hat{\mathcal{X}}$ in equation (7). The duality between $C_{10}$ and $R_{01}$ has been shown in [1], [2], [3] and [4]. The general duality between $C_{ij}$ and $R_{ji}$ for all four cases is shown in [5], together with the incremental duality in the Gaussian case.

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## II. Unification and Extension

We unify the four special cases of channel capacity with state information with one formula [6] that may look like the odd case at first. This unification also extends to the case of having i.i.d. pairs of state information $(S_1, S_2)$ to the sender and the receiver, respectively. Let the state information $(S_{1i}, S_{2i})$ i.i.d $\sim p(s_1, s_2)$, with $S_1^n$ available to the sender, and $S_2^n$ to the receiver. We have a memoryless channel $p(y|x, s_1, s_2)$. The encoder is of the form $X^n(W, S_1^n)$ and the decoder is of the form $\hat{W}(Y^n, S_2^n)$.

**Theorem 1** *The capacity of this channel is*

$$C = \max_{p(u,x|s_1)} [I(U; (S_2, Y)) - I(U; S_1)] \tag{9}$$

It can be verified that all four special cases (equations (1) to (4)) are corollaries of the above theorem. This general problem setup also highlights the issue of nested state information and the tradeoff between channel capacity and state estimation.

We also show that four special cases of rate distortion with state information can also be unified with the following Wyner-Ziv formula dual to equation (9).

$$R(D) = \min_{p(u|x,s_1)p(\hat{x}|u,s_2)} [I(U; (S_1, X)) - I(U; S_2)] \tag{10}$$

under the distortion constraint $\sum_{x,u,s_1,s_2} d(x,\hat{x}) p(u|x,s_1) p(\hat{x}|u,s_2) p(x,s_1,s_2) \leq D$. Together with the connections with other multiterminal data compression problems, the above result shows that the Wyner-Ziv formula unifies different versions of distributed data compression, lossless or lossy, with one way helper information.

Thus the main point is that all 8 results in equations (1) to (8) are given by the common form $I(U;Y) - I(U;S)$.

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