Growth optimal policies with transaction costs

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Abstract — **In this work, we address the problem of growth optimal investment. This problem was introduced in i.i.d. markets by Breiman [1]. Since then it has been extended to stationary ergodic markets markets and also to markets with no distributional assumptions [2],[3],[4].**

The previous work assumes that the market has no transaction costs, i.e. the investor is allowed to trade at no cost. Constant rebalanced strategies that are optimal in the frictionless case would lead to immediate bankruptcy in markets with costs. We extend the growth optimal framework to markets with costs.

I. Introduction

Suppose an investor starts with an initial wealth of $S_0 = 1$ invested in a portfolio $\mathbf{b}_1 = (\mathbf{b}_1(i))_{(1\leq i\leq m)} \in \mathcal{S} = \{\mathbf{b} : \mathbf{b}(i) \geq 0, \sum_{i=1}^m \mathbf{b}(i) = 1\}$. The sequence of price relatives of the assets $\{\mathbf{X}_n = (\mathbf{X}_n(i))_{(1\leq i\leq m)}\}_{n\in \mathbf{Z}^+}$ is assumed to be i.i.d. non-negative random vectors in $\mathbf{R}^m_+$. At the beginning of each period n, the investor who is at the portfolio $\mathbf{b}_n$ has the option to trade to a new portfolio $\mathbf{z}_n$ to position himself better to reap the market gains..

This move results in a loss of wealth because of transaction costs. We assume that the i-th asset incurs a proportional transaction cost at the rate λ_i. Since transaction costs are proportional, if $w(\mathbf{z}_n, \mathbf{b}_n)$ is the net wealth when one dollar is traded from portfolio $\mathbf{b}_n$ to portfolio $\mathbf{z}_n$, then S_{n-1} invested at portfolio $\mathbf{b}_n$ nets $w(\mathbf{z}_n, \mathbf{b}_n)S_{n-1}$ at portfolio $\mathbf{z}_n$.

Therefore at the end of the period n, the total wealth S_n is given by

$$S_n = (\mathbf{z}_n^t \mathbf{X}_n) w(\mathbf{z}_n, \mathbf{b}_n) S_{n-1} = \prod_{1\leq k\leq n} (\mathbf{z}_k^t \mathbf{X}_k) w(\mathbf{z}_k, \mathbf{b}_k).$$

A feasible policy ρ for the initial position $\mathbf{b}_1$ is a collection of functions $\rho = \{\rho_n\}_{(n\geq 1)}$, $\rho_n : \mathcal{S} \to \mathcal{S}$, with the interpretatoin that at time n the investor rebalances to $\rho_n(\mathbf{b}_n)$.

In this work we are interested in maximizing the growth rate of wealth S_n, i.e. in characterizing the *growth rate function*

$$d(\mathbf{b}) = \sup_{\rho} \liminf_{n\to\infty} \frac{1}{n} \mathbf{E}\left(\log(S_n^{\rho}(\mathbf{b}))\right).$$

where $S_n^{\rho}(\mathbf{b})$ is the wealth generated at time n by a policy ρ starting from the portfolio $\mathbf{b}$.

II. Results

For the general i.i.d. market we show that the growth rate function $d(\mathbf{b})$ is a constant d on the simplex $\mathcal{S}$. We characterize this growth rate by an implicit recursion equation and establish that the problem has a stationary Markov optimal policy. Further we show that there is an optimal stationary Markov policy ρ^* of the ***control-limit*** type. This policy has an associated compact, connected ***no-trade*** region $\mathcal{N}$. When the portfolio strays outside $\mathcal{N}$ the policy ρ^* corrects it to the boundary $\partial\mathcal{N}$. The no-trade region $\mathcal{N}$ is a cone in the gradient space.

We consider two extensions of this problem. The first is a two-asset market with no distributional assumptions on the market realization. In two asset markets the control-limit policies are interval policies, i.e. the policy corrects the portfolio to an interval. We exhibit a sequential investment strategy that achieves the same wealth, to first order in the exponent, as the best interval strategy in hindsight, i.e. this ***universal*** investment strategy induces the same growth rate of wealth as that achieved by the best interval strategy in hindsight. It follows that if the market is i.i.d., then the universal policy achieves the optimal growth rate for the unknown distribution.

Next we consider horse race markets. In horse race markets, one of the assets (horses) pays off and all the other assets (horses) go broke. If p_i is the probability that the i-th horse wins, paying o_i odds, then the optimal growth rate for this market is given by

$$d = \sum_{i=1}^m p_i \log(o_i) - H(p) + \sum_{i=1}^m p_i(1-p_i)\log(1-\lambda_i^2),$$

where $H(p)$ is the entropy of $(p_1, \ldots, p_m)$.

If the investor, instead of knowing the probability p_i, has only an estimate q_i, then the achievable growth rate is given by

$$d_q = d - D(p\|q),$$

where $D(p\|q)$ is the relative entropy p and q.

In case the investor has access to side information $\{\mathbf{Y}_k\}$, where $\{(\mathbf{X}_k, \mathbf{Y}_k)\}_{(k\geq 1)}$ are i.i.d., and invests accordingly, then the rate d_y with side information is given by

$$d_y = d + I(\mathbf{X}, \mathbf{Y}),$$

where $I(\mathbf{X}, \mathbf{Y})$ is the mutual information between $\mathbf{X}$ and $\mathbf{Y}$. Finally we show that there is a universal investment strategy that achieves the same growth rate as the best rebalanced strategy in hindsight.

References

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