[13] H. K. Thapar and A. Patel, "A class of partial response systems for increasing storage densities in magnetic recording," *IEEE Trans. Magn.*, vol. MAG-23, pp. 3666–3668, Sept. 1987.
[14] J. K. Wolf, "Permutation codes, $(d, k)$ codes and magnetic recording," in *Proc. 1990 IEEE Colloq. South America, Argentina, Brazil, Chile, Uruguay*, Sept. 1990.
[15] R. Blahut, *Digital Transmission of Information*. Englewood Cliffs, NJ: Prentice-Hall, 1991.
[16] R. Karabed and P. Siegel, "Coding for higher order partial response channels," in *Proc. SPIE'95*, pp. 115–126.
[17] B. Moision, P. Siegel, and E. Soljanin, "Distance-enhancing codes for digital recording," *IEEE Trans. Magn.*, vol. 34, pp. 69–74, Jan. 1998.
[18] J. W. Bergmans, *Digital Baseband Transmission and Recording*. Boston, MA: Kluwer, 1996.

# Growth Optimal Investment in Horse Race Markets with Costs

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***Abstract*—We formulate the problem of growth optimal investment in horse race markets with proportional costs and study growth optimal strategies both for stochastic horse races as well as races where one does not make any distributional assumptions. Our results extend all known results for frictionless horse race markets to their natural analog in markets with costs.**

***Index Terms*—Growth optimality, proportional transaction costs, universal investment.**

## I. Introduction

In this work we formulate the problem of growth optimal investment in horse race markets with proportional transaction costs. These markets are, as the name suggests, very much like horse races, i.e., in every market period one of the assets pays off and all the other assets pay nothing. In fact, the wealth invested in these assets is lost completely. Horse race markets, also called erodible asset markets, are very special cases of general markets and are in a sense the extreme points for the distribution of asset returns [1], [2].

The objective of growth optimal investment is to maximize the long-run interest rate. Growth optimal investment in independent and identically distributed (i.i.d.) horse race markets with no transaction costs was introduced by Kelly [3]. Kelly showed that log-optimum investment, where the investor maximizes the conditional expected logarithm of his one-step return, maximizes the growth rate of the cumulative wealth. Kelly established a duality between the entropy rate and the growth rate by showing that they sum to a constant. Breiman [4] extended this framework to investment in i.i.d. markets with general asset returns. Algoet and Cover [5] showed that conditionally log-optimum investment is, indeed, growth optimal for all frictionless stationary ergodic markets with general asset returns. This work (see also [6]) extended the duality between information rate and growth rate to stationary ergodic markets. Subsequently, Cover [7] introduced universal investment where one does not make any distributional assumptions on the sequence of asset returns and constructed a sequential policy that does as well in growth rate as any other constant rebalanced policy, even those chosen in hindsight. The universal result was extended to an individual sequence result by Ordentlich and Cover [2], [8].

The preceding work assumes that transactions in the market do not incur any costs. Unfortunately, the policies designed for frictionless markets do not perform well in a market with costs. In fact, in a continuous time market such policies lead to immediate bankruptcy [9]. The situation in discrete time markets is not as severe [10], but one can do substantially better by incorporating the costs into the model. Growth optimal investment in markets with costs was introduced by Taksar *et al.* [11] in the context of geometric Wiener markets with one risky asset and cash. Several related works, notably Davis *et al.* [9] and Akian *et al.* [12], study investment policies that maximize the discounted utility of the consumption stream in markets with costs. Iyengar [13] and Akian *et al.* [14] have recently extended continuous time growth optimal investment to the general case of several risky assets.

The outline of this correspondence is as follows. In Section II we introduce our model for a market with proportional transaction costs and then formulate the problem of growth optimal investment in discrete-time stochastic horse race markets. We show that the conditionally log-optimum policy is growth-optimal. We then discuss some properties of the log-optimum policy and its sensitivity to the information structure. Although the results in this section are specific to the special case of horse race markets, we believe that the intuition developed here will generalize to general discrete-time markets [15]. Section III studies universal investment in horse race markets with proportional transaction costs. The last section includes some concluding remarks, connection to general markets, and some open issues.

## II. Investment in Repeated Stochastic Horse Races

A horse race market is characterized by a sequence $\{X_n : n \geq 1\}$ of price relative vectors independently and identically distributed as follows:

$$X_n = [\underbrace{0, \ldots, 0}_{i-1}, o_i, \underbrace{0, \ldots, 0}_{m-i}]^t = o_i e_i$$

with probability $p_i, i = 1, \ldots, m$, where $e_i$ denotes the $i$th basis vector. Thus only one of the assets (horses) pays off $o_i$ for 1 odds and all the other $m-1$ assets pay nothing. Although we make the assumption that the sequence $\{X_n : n \geq 1\}$ is i.i.d., we show later in this section that the results easily generalize to include stationary, ergodic horse race markets.

The market is assumed to have proportional transaction costs, i.e., every transaction incurs a cost proportional to the dollar amount of the transaction. The rate of proportional cost levied on asset $i$ is denoted by $\lambda_i$, i.e., the sale of 1 dollar's worth of asset $i$ nets only $(1-\lambda_i)$ dollars; and similarly, the purchase of 1 dollar's worth of asset $i$ costs $(1+\lambda_i)$ dollars. (Although we assume that the costs are symmetric, the analysis extends to the asymmetric case almost as is.) We assume that cash is the intermediary in all transactions. Therefore, if the investor wants to move one dollar from asset $i$ to asset $j$, he must first sell a dollar's worth of asset $i$ to get $(1-\lambda_i)$ assets in cash and then reinvest this amount in asset $j$ to net $(1-\lambda_i)/(1+\lambda_j)$ dollars worth of asset $j$. Notice that although the structure of market returns is that of horse races, we retain the structure of transaction costs of general stock markets. We

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take this approach since we are interested in analyzing horse races as an instance of the general market with the hope that the understanding will translate to the general case.

A portfolio vector $b = (b(1), b(2), \ldots, b(m)), \sum_{i=1}^m b(i) = 1$, represents the proportion of the wealth in the various assets. We allow only long positions in the assets, which requires that the proportions $b(i) \geq 0$, for all $i = 1, \ldots, m$. We will denote the space of allowed portfolios by $\mathcal{B}$, i.e.,

$$\mathcal{B} = \left\{ b \in \boldsymbol{R}^m \,\middle|\, b(i) \geq 0, \sum_{i=1}^m b(i) = 1 \right\}.$$

The investor begins with an initial wealth $S_1 = 1$, which he invests in the portfolio $b_1$. As a consequence of the transaction costs, the net wealth of the investor in the portfolio $b_1$ is less than the initial dollar. We denote the resulting wealth by $w(b_1)$. Since investing $w(b_1)b_1(i)$ dollars in asset $i$ costs the investor $(1+\lambda_i)w(b_1)b_1(i)$, it follows that the net wealth $w(b_1)$ is given by the equation

$$w(b_1) = \frac{1}{\sum_{i=1}^m (1+\lambda_i) b_1(i)}.$$

To be specific, the portfolio of the investor is $b_1 \in \mathcal{B}$, the dollar value of the investment is $w(b_1)$, and the dollar holdings are $(w(b_1)b_1(1), \ldots, w(b_1)b_1(m)) \in \boldsymbol{R}_+^m$.

Subsequently, the market reveals the price relative vector $X_1$, and the $w(b_1)b_1(i)$ dollars invested in asset $i$ is now worth $w(b_1)b_1(i)X_1(i)$. As a result, the net wealth $S_2$ at the beginning of the second market period is

$$S_2 = \sum_{i=1}^m w(b_1) b_1(i) X_1(i) = w(b_1) b_1^t X_1$$

and the portfolio $z_2 \in \mathcal{B}$ of the investor at the beginning of the second market period is given by

$$z_2(i) = \frac{w(b_1)b_1(i)X_1(i)}{S_2} = \frac{b_1(i)X_1(i)}{b_1^t X_1}, \qquad i = 1, \ldots, m.$$

Since only one of the components of $X_1$ is nonzero, we can express the portfolio $z_2$ more simply as

$$z_2 = \frac{X_1}{\sum_{i=1}^m X_1(i)}.$$

Therefore, the portfolio of the investor at the beginning of the second period is

$$z_2 = e_i, \text{ with probability } p_i, \qquad i = 1, \ldots, m.$$

The investor is now allowed to readjust his portfolio to a new portfolio $b_2$ before the price relative vector $X_2$ is revealed. (In contrast with real horse races we do not require that the investor "cash out" his holdings in the winning horse and reinvest in the horses.) This process, then, repeats itself in every market period.

Notice that the first investment period is distinguished in that the investor has wealth $S_1 = 1$, which he distributes freely over the available assets. In the subsequent investment periods the investor has his wealth tied up in one asset (the asset that "won" in the previous period) and wishes to redistribute his wealth.

At the beginning of the $n$th market period, the net wealth of the investor is denoted by $S_n$, and the portfolio of the investor is denoted by $z_n$, where

$$z_n = \frac{X_{n-1}}{\sum_{i=1}^m X_{n-1}(i)} = e_i,$$

with probability $p_i, i = 1, \ldots, m$.

We will denote the readjusted portfolio at time $n$ by $b_n$. The readjustment from $z_n$ to $b_n$ results in a loss of wealth because of the transaction costs. If the investor with wealth $S_n$ in portfolio $z_n$ reinvests his wealth in a new portfolio $b_n$, then the net wealth will drop from $S_n$ to $w(b_n, z_n)S_n$, where the factor $w(b_n, z_n)$ is given by the nonnegative solution $w$ of the equation

$$1 = w + \sum_{i=1}^m \lambda_i |w b_n(i) - z_n(i)|. \tag{1}$$

Equation (1) equates the initial wealth to the sum of the final wealth $w$ and the costs $\sum_i \lambda_i |w b_n(i) - z_n(i)|$ incurred in the trade. Since the transaction costs are proportional, the net wealth invested in portfolio $b_n$ is given by $w(b_n, z_n)S_n$.

Thus at the beginning of the period $n+1$, the total wealth $S_{n+1}$ is given by

$$S_{n+1} = w(b_n, z_n) \left( b_n^t X_n \right) S_n. \tag{2}$$

Since the initial wealth $S_1 = 1$, the compounded wealth $S_n$ is given by

$$S_n = \prod_{k=1}^{n-1} w(b_k, z_k) \left( b_k^t X_k \right) \tag{3}$$

where we define $w(b_1, z_1) = w(b_1)$ for notational uniformity.

In this work we are interested in maximizing the growth rate $g$ of the wealth $S_n$, where $g$ is defined as

$$\begin{aligned} g &= \liminf_{n\to\infty} \frac{1}{n} \boldsymbol{E} \log S_n \\ &= \liminf_{n\to\infty} \left\{ \frac{1}{n} \sum_{k=1}^{n-1} \boldsymbol{E} \log \left( w(b_k, z_k) b_k^t X_k \right) \right\}. \end{aligned} \tag{4}$$

We restrict the investor to policies that are nonanticipating and self-financing, i.e., all transaction costs and future investments are financed by the wealth generated by the policy without any fresh wealth. We allow the investment policy to be arbitrarily dependent on the past portfolios, past investment decisions, and the past realizations of the market. Notice that we do not restrict the investor from using randomization or any prediction about the future that he might possess. The set of all such policies, the admissible policies, is denoted by $\Pi$.

Given this framework, the optimization problem is as follows. Characterize the optimal growth rate $g$ given by

$$g = \sup_{\rho \in \Pi} \left\{ \liminf_{n\to\infty} \frac{1}{n} \boldsymbol{E} \log S_n^\rho \right\} \tag{5}$$

where $S_n^\rho$ is the total wealth generated by policy $\rho$ up to time $n$. And if the supremum in (5) is achieved, then characterize a policy $\rho_{\rm opt}$ that achieves $g$, i.e., a policy that is growth-optimal.

Extending the ideas of Kelly [3] and Algoet and Cover [5], we define the conditionally log-optimum policy $\rho^*$ as follows. At time $n = 1$, the policy $\rho^*$ invests the initial dollar in the portfolio

$$b_1^* = \operatorname*{argmax}_{b \in \mathcal{B}} \boldsymbol{E} \log(w(b, z_1) b^t X). \tag{6}$$

At the beginning of each subsequent period $n \geq 2$, the policy $\rho^*$ corrects the portfolio of the investor to

$$b_n^* = \operatorname*{argmax}_{b \in \mathcal{B}} \boldsymbol{E}\{\log(w(b, z_n)b^t X) \mid z_n\} \tag{7}$$

i.e., the portfolio choice $b_n^*$ at time $n$ is a function of the portfolio $z_n$ at the beginning of the market period. We will denote the wealth stream associated with $\rho^*$ by $S_n^*$. Since the decision at time $n$ is completely determined by the portfolio $z_n$, the policy $\rho^*$ is Markov and nonanticipating. By the definition of $w(b_n, z_n)$, it follows that the policy is self-financing. Thus $\rho^*$ is an admissible policy.

A more explicit description of $\rho^*$ is as follows. Suppose $z_n = e_i$, i.e., the $i$th asset "wins" in the $(n-1)$th market period. Since all the wealth is invested in asset $i$ the investment decision consists of selling a certain fraction of the holdings in asset $i$ and then reinvesting the resulting cash in the other assets. Suppose the investor sells a fraction $(1 - q_i)$ of his wealth asset $i$ and invest the proceeds $(1 - \lambda_i)(1 - q_i)$ into the other assets. If the fraction invested in horse $j$ is $(1 - \lambda_i)q_j$, then the self-financing constraint implies that

$$\sum_{j \neq i} (1 - \lambda_i) q_j = (1 - \lambda_i)(1 - q_i). \tag{8}$$

Thus the fraction $w(b, e_i)b(j)$ invested in asset $j$ is given by

$$w(b, e_i)b(j) = \begin{cases} q_i, & j = i \\ \frac{(1-\lambda_i)}{(1+\lambda_j)}(1 - q_i), & i \neq j. \end{cases}$$

For a given $q_i$, a generic term in the sum in (4) can be rewritten as follows:

$$\begin{aligned} \boldsymbol{E} \log(w(b, e_i)b^t X) &= p_i \log(q_i o_i) \\ &\quad + \sum_{j \neq i} p_j \log\left(\left(\frac{1-\lambda_i}{1+\lambda_j}\right) q_j o_j\right) \\ &= \sum_{j=1}^m p_j \log(o_j) + \sum_{j=1}^m p_j \log(q_j) \\ &\quad + \sum_{j \neq i} p_j \log\left(\frac{1-\lambda_i}{1+\lambda_j}\right) \\ &= \sum_{j=1}^m p_j \log(o_j) - H(p) - D(p \| q) \\ &\quad + \sum_{j \neq i} p_j \log\left(\frac{1-\lambda_i}{1+\lambda_j}\right) \end{aligned} \tag{9}$$

where

$$H(p) = -\sum_{i=1}^m p_i \log(p_i)$$

is the entropy of $p$, and

$$D(p \| q) = \sum_{i=1}^m p_i \log(p_i / q_i)$$

is the relative entropy between $p$ and $q$. (From (8) we have that $q \in \mathcal{B}$ and, therefore, can be interpreted as a probability mass function.)

Since $D(p \| q) \geq 0$, with equality if and only if $p = q$, it follows that the optimal choice for $q$ in (9) is $p$. Working backward from the definition of $q$, we see that the dollar amounts invested in the assets $j \neq i$ after asset $i$ "wins," i.e., $z_n = e_i$, is $(1 - \lambda_i)p_j$. Therefore, the optimal portfolio $b_i^* \in \boldsymbol{R}_+^m$ when $z_n = e_i$ is

$$b_i^*(j) = \begin{cases} \dfrac{p_i}{p_i + \sum_{l \neq i} \left(\frac{1-\lambda_i}{1+\lambda_l}\right) p_l}, & j = i, \\[2ex] \dfrac{\left(\frac{1-\lambda_i}{1+\lambda_j}\right) p_j}{p_i + \sum_{l \neq i} \left(\frac{1-\lambda_i}{1+\lambda_l}\right) p_l}, & j \neq i, \end{cases} \qquad j = 1, \ldots, m. \tag{10}$$

A similar analysis shows that the optimal portfolio $b_1^*$ at time $n = 1$ is given by

$$b_1^* = p. \tag{11}$$

The policy $\rho^*$ has the following simple characterization. If asset $i$ wins, keep a proportion $p_i$ of it and sell $(1 - p_i)$. Take the proceeds $(1 - \lambda_i)(1 - p_i)$ of the sale and invest it in the remaining assets in the proportions $(p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_m)$, i.e., invest $(1 - \lambda_i)p_j$ in asset $j \neq i$. As a result, the final wealth is distributed according to

$$(\gamma_1 p_1, \ldots, \gamma_{i-1} p_{i-1}, p_i, \gamma_{i+1} p_{i+1}, \ldots, \gamma_m p_m)$$

where $\gamma_j = (1 - \lambda_i)/(1 + \lambda_j)$. Thus the intent is always the same—place sell and buy orders as if there were no transactions costs [3] and accept whatever you get as a result.

Since $\{X_n : n \geq 1\}$ is assumed to be i.i.d., it follows from (9) that the growth rate $g^*$ associated with the policy $\rho^*$ is given by

$$\begin{aligned} g^* &= \sum_{i=1}^m p_i \left[ \sum_{j=1}^m p_j \log(o_j) - H(p) + \sum_{j \neq i} p_j \log\left(\frac{1-\lambda_i}{1+\lambda_j}\right) \right] \\ &= \sum_{j=1}^m p_j \log(o_j) - H(p) + \sum_{i=1}^m p_i (1 - p_i) \log\left(\frac{1-\lambda_i}{1+\lambda_i}\right). \end{aligned} \tag{12}$$

Thus

$$g^* = g_0 + \sum_{i=1}^m p_i (1 - p_i) \log\left(\frac{1-\lambda_i}{1+\lambda_i}\right)$$

where $g_0$ is the growth rate in frictionless markets.

The first result of this correspondence states that the conditional log-optimum policy $\rho^*$ is growth-optimal.

*Theorem 1:* In a horse race market with $m$ assets, where asset $i$ pays $o_i$ for 1 odds with probability $p_i$, the optimal growth rate of wealth $g$ is given by

$$g = \sum_{i=1}^m p_i \log(o_i) - H(p) + \sum_{i=1}^m p_i (1 - p_i) \log\left(\frac{1-\lambda_i}{1+\lambda_i}\right) \tag{13}$$

where

$$H(p) = -\sum_{i=1}^m p_i \log(p_i)$$

is the entropy of the probability mass function $p = (p_1, \ldots, p_m)$. Moreover, the conditionally log-optimum policy $\rho^*$ described in (10) and (11) is growth-optimal.

*Proof:* Let $\rho^*$ be the conditionally log-optimum policy defined by (10) and (11). Let $\rho$ be any other admissible policy. Let $\{S_n^\rho : n \geq 1\}$ and $\{b_n^\rho : n \geq 1\}$ be the corresponding wealth stream and portfolio choices, respectively.

For any policy $\rho \in \Pi$, the sequence of market opening portfolios $\{z_n^\rho\}$ is given by

$$z_n^\rho = \frac{X_{n-1}}{\sum_{i=1}^m X_{n-1}(i)} = z_n^*, \qquad \forall n \geq 2$$

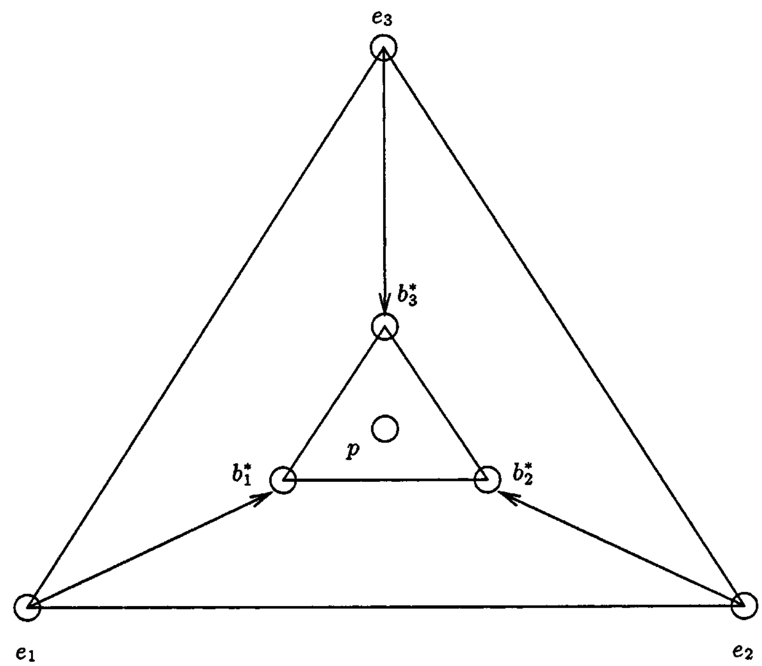


Fig. 1. Optimum policy with transaction costs.

where $\{z_n^* : n \geq 2\}$ are the sequence of market opening portfolios corresponding to the conditionally log-optimum policy $\rho^*$.

Therefore, $z_n^\rho = z_n^*$ for all policies $\rho \in \Pi$ on a sample-path-by-sample-path basis. In the rest of the proof, we will denote the sequence of market opening portfolios by $\{z_n\}$.

Let $\boldsymbol{P}_{z_n}^\rho$ be the conditional distribution of $b_n^\rho$ given $z_n$. Since the policy $\rho$ is nonanticipating, we have

$$\begin{aligned}\boldsymbol{E}&\left\{\log\left(w\left(b_n^\rho, z_n\right)\left(b_n^\rho\right)^t X_n\right)\Big| z_n\right\}\\ &= \int_{\mathcal{B}} \boldsymbol{E}\log\left(w(b, z_n) b^t X\right) \boldsymbol{P}_{z_n}^\rho(db),\\ &\leq \int_{\mathcal{B}} \boldsymbol{E}\log\left(w\left(b_{z_n}^*, z_n\right)\left(b_{z_n}^*\right)^t X\right) \boldsymbol{P}_{z_n}^\rho(db),\\ &= \boldsymbol{E}\log\left(w(b_{z_n}^*, z_n)\left(b_{z_n}^*\right)^t X\right),\\ &= \boldsymbol{E}\{\log(w(b_n^*, z_n)(b_n^*)^t X_n) \mid z_n\} \end{aligned} \tag{14}$$

where (14) follows from the definition of the conditionally log-optimum policy $\rho^*$.

Since $S_{n+1}^\rho = (w(b_n^\rho, z_n)(b_n^\rho)^t X_n) S_n^\rho$, it follows that

$$\boldsymbol{E}\left\{\log\left(\frac{S_{n+1}^*}{S_n^*}\right)\Bigg| z_n\right\} \geq \boldsymbol{E}\left\{\log\left(\frac{S_{n+1}^\rho}{S_n^\rho}\right)\Bigg| z_n\right\}.$$

On unconditioning we have

$$\boldsymbol{E}\left\{\log\left(\frac{S_{n+1}^*}{S_n^*}\right)\right\} \geq \boldsymbol{E}\left\{\log\left(\frac{S_{n+1}^\rho}{S_n^\rho}\right)\right\}.$$

Since the initial wealth $S_1 = S_1^* = 1$, we have

$$\begin{aligned}\boldsymbol{E}\log S_n^* &= \boldsymbol{E}\left\{\sum_{k=1}^{n-1}\log\left(\frac{S_{k+1}^*}{S_k^*}\right)\right\}\\ &\geq \boldsymbol{E}\left\{\sum_{k=1}^{n-1}\log\left(\frac{S_{k+1}^\rho}{S_k^\rho}\right)\right\} = \boldsymbol{E}\log S_n^\rho.\end{aligned}$$

Therefore,

$$\liminf_{n\to\infty}\frac{1}{n}\boldsymbol{E}\log\left(\frac{S_n^*}{S_n}\right) \geq 0.$$

From (12) it follows that the growth rate is

$$g = g^* = \sum_{i=1}^m p_i\log(o_i) - H(p) + \sum_{i=1}^m p_i(1-p_i)\log\left(\frac{1-\lambda_i}{1+\lambda_i}\right). \quad \square$$

It is easy to show that the probability mass function $p$ lies in the convex hull of the the optimal portfolio choices $b_i^*$ defined in (10). Therefore, the optimal policy no longer corrects the investor's portfolio to the optimal frictionless portfolio $p$ [3]; instead, it corrects the portfolio to the "nearest" portfolio on the boundary of a set that contains the optimal frictionless portfolio $p$ (see Fig. 1). This interpretation of the optimal policy extends to general discrete-time markets [15].

Moreover, from the characterization of the growth rate $g$, given in (13), the duality between growth rate and the entropy rate $H(p)$ is apparent; costs simply decrease the growth rate by a constant. Thus the loss $l(\lambda)$ due to transactions costs is given by

$$l(\lambda) = \sum_{i=1}^m p_i(1-p_i)\log\left(\frac{1-\lambda_i}{1+\lambda_i}\right). \tag{15}$$

In the limit of small transaction costs

$$l(\lambda) \approx -2\sum_{i=1}^m p_i(1-p_i)\lambda_i.$$

(This asymptotic result is different from the asymptotic result for continuous time markets [13], [16].)

Unfortunately, the above proof technique does not extend to general markets with costs. The crux of the proof is that the sequence of market opening portfolios $\{z_n : n \geq 1\}$ is the same for all policies $\rho \in \Pi$—one and only one asset "wins," and which asset "wins" is not a function of the bets put on it! This is not true in general markets. We present the solution for the general i.i.d. market in [15].

*Example 1:* Suppose the market consists of three assets paying 6 for 1 odds, each of which "wins" with equal probability, i.e., $p_i = \frac{1}{3}$, $i = 1, 2, 3$. Suppose also that the transaction cost $\lambda_i = 0.2$ for all $i$.

Then the optimal portfolio choice $b_1^*$ is given by

$$b_1^* = \frac{1}{\frac{1}{3} + \frac{2}{3}\left(\frac{1-0.2}{1+0.2}\right)} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3}\left(\frac{1-0.2}{1+0.2}\right) \\ \frac{1}{3}\left(\frac{1-0.2}{1+0.2}\right) \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{2}{7} \\ \frac{2}{7} \end{pmatrix}.$$

Similarly, the portfolio $b_2^*$ and $b_3^*$ are given by

$$b_2^* = \begin{pmatrix} \frac{2}{7} \\ \frac{3}{7} \\ \frac{2}{7} \end{pmatrix} \qquad b_3^* = \begin{pmatrix} \frac{2}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{pmatrix}.$$

The corresponding growth rate $g^*$ is given by

$$\begin{aligned} g^* &= \sum_{i=1}^{3} p_i \log(p_i o_i) + \sum_{i=1}^{3} p_i(1-p_i)\log\left(\frac{1-\lambda_i}{1+\lambda_i}\right) \\ &= \frac{2}{3}\log\left(\frac{2}{3}\right) + 1. \end{aligned}$$

Note that

$$p = \frac{1}{3}b_1^* + \frac{1}{3}b_2^* + \frac{1}{3}b_3^*.$$

Thus $p$ is in the convex hull of $b_i^*$, as previously noted. □

From the explicit characterization of the policy $\rho^*$ and the growth rate $g$, several results follow.

*Theorem 2:* Under the conditions of Theorem 1, the log-optimum policy $\rho^*$ achieves the growth rate $g$ in an almost sure sense, i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\log S_n^* = g, \qquad \text{with probability } 1$$

where $\{S_n^* : n \geq 1\}$ is the wealth generated by $\rho^*$.

*Proof:* Let $n_{ij}$ be the number of times the pair $(e_i, e_j)$ appears in the first $n$ terms of the sequence $\{z_k : k \geq 1\}$. Then

$$\log S_n^* = \sum_{ij} n_{ij} \log(w(b_i^*, e_i) b_i^*(j) o_j).$$

Since $\{z_k : k \geq 1\}$ is i.i.d., it follows that for all $i, j$

$$\lim_{n\to\infty} \frac{n_{ij}}{n} = p_{ij} = p_i p_j, \qquad \text{with probability } 1.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}\log S_n^* &= \lim_{n\to\infty} \sum_{i,j=1}^{m} \frac{n_{ij}}{n} \log(w(b_i^*, e_i) b_i^*(j) o_j), \\ &= \sum_{i,j=1}^{m} p_i p_j \log(w(b_i^*, e_i) b_i^*(j) o_j), \\ &= g. \end{aligned}$$ □

The next result establishes that the policy $\rho^*$ is optimal in an almost sure sense as well.

*Theorem 3:* Let $\{S_n^* : n \geq 0\}$ be the wealth generated by the optimal policy $\rho^*$ and $\{S_n^\rho : n \geq 0\}$ be the wealth stream generated by any admissible policy $\rho$, then

$$\limsup_{n\to\infty} \frac{1}{n}\log\left(\frac{S_n^\rho}{S_n^*}\right) \leq 0, \qquad \text{with probability } 1.$$

*Proof:* Let $\{b_n : n \geq 1\}$ be the sequence of portfolios generated by the any policy $\rho$ and $\{b_n^* : n \geq 1\}$ be the sequence corresponding to the conditionally log-optimum policy $\rho^*$. By construction we have

$$\begin{aligned} \boldsymbol{E}\log(w(b_1^*)(b_1^*)^t X) &= \max_{b\in\mathcal{B}} \boldsymbol{E}\log(w(b) b^t X) \\ &\geq \boldsymbol{E}\log(w(b_1)(b_1)^t X), \end{aligned}$$

and for all $i = 1, \ldots, m$

$$\begin{aligned} &\boldsymbol{E}\{\log w(b_n^*, z_n)(b_n^*)^t X \mid z_n = e_i\} \\ &\quad = \max_{b\in\mathcal{B}} \boldsymbol{E}\{\log w(b, z_n) b^t X \mid z_n = e_i\} \\ &\quad \geq \boldsymbol{E}\left\{\log w(b_n, z_n) b_n^t X \,\middle|\, z_n = e_i\right\}. \end{aligned}$$

Therefore, for all $n \geq 1$

$$\boldsymbol{E}\log\left(\frac{w(b_n, z_n) b_n^t X}{w(b_n^*, z_n)(b_n^*)^t X}\right) \leq 1. \tag{16}$$

Since $S_1^\rho = 1$ and

$$S_n^\rho = \prod_{k=1}^{n-1} w(b_k, z_k)(b_k^\rho)^t X_k$$

we have that for all $n \geq 1$

$$\boldsymbol{E}\log\left(\frac{S_n^\rho}{S_n^*}\right) \leq 0. \tag{17}$$

We next establish that (17) implies that $\boldsymbol{E}(S_n^\rho / S_n^*) \leq 1$, for all $n \geq 1$. The proof is identical to proofs of [1, Theorem 2] and [5, Theorem 1 ].

Suppose there exists an admissible policy $\rho \in \Pi$ and a time instant $n$ such that

$$\boldsymbol{E}\left(\frac{S_n^\rho}{S_n^*}\right) > 1. \tag{18}$$

Let $\rho_\lambda$ be the policy that divides the initial wealth $S_1 = 1$ into an amount $\lambda(>0)$ invested according to the policy $\rho$ and an amount $(1-\lambda)(>0)$ invested according to the policy $\rho^*$, pooling the money "on paper" only at time $n$. Then, the wealth $S_n^\lambda$, corresponding to the policy $\rho_\lambda$, is given by

$$S_n^\lambda = \lambda S_n^\rho + (1-\lambda) S_n^*.$$

The policy $\rho_\lambda$ is clearly admissible and from (18) it follows that $\boldsymbol{E}(S_n^\lambda / S_n^*) > 1$. We will now derive a contradiction by establishing that $\boldsymbol{E}\log(S_n^\lambda / S_n^*) > 0$.

Define $Y = (S_n^\lambda / S_n^*) - 1$ and $Y_c = \min\{Y, c\}$. Since, by hypothesis, $\boldsymbol{E}Y > 0$, there exists a real number $c_0 \geq 2$ such that $\boldsymbol{E}Y_{c_0} > 0$. From a Taylor series expansion it follows that

$$\log\left(S_n^\lambda / S_n^*\right) \geq \log(1 + \lambda Y_{c_0}) = \lambda Y_{c_0} - \frac{a^2}{2}$$

for some $a$ between $\lambda$ and $\lambda Y_{c_0}$. Since $Y_{c_0} \leq c_0$, we have that

$$\log\left(S_n^\lambda / S_n^*\right) \geq \lambda Y_{c_0} - \frac{\lambda^2 c_0^2}{2}.$$

Finally, since $\boldsymbol{E}Y_{c_0} > 0$, it is possible to choose $\lambda$ sufficiently small so that

$$\boldsymbol{E}\log\left(\frac{S_n^\rho}{S_n^*}\right) > 0.$$

This contradicts (17), thereby proving that for all $\rho \in \Pi$, and all $n \geq 1$

$$\boldsymbol{E}\left(\frac{S_n^\rho}{S_n^*}\right) \leq 1. \tag{19}$$

By Markov's inequality, we have for all $\epsilon > 0$

$$\boldsymbol{P}\left\{\frac{1}{n}\log\left(\frac{S_n^\rho}{S_n^*}\right) > \epsilon\right\} = \boldsymbol{P}\left\{\frac{S_n^\rho}{S_n^*} > e^{\epsilon n}\right\} \le e^{-\epsilon n}\boldsymbol{E}\left(\frac{S_n^\rho}{S_n^*}\right) \le e^{-\epsilon n}.$$

Therefore,

$$\sum_{n=1}^{\infty}\boldsymbol{P}\left\{\frac{1}{n}\log\left(\frac{S_n^\rho}{S_n^*}\right) > \epsilon\right\} \le \sum_{n=1}^{\infty} e^{-\epsilon n} < \infty.$$

By the Borel–Cantelli Lemma, it follows that

$$\boldsymbol{P}\left\{\frac{1}{n}\log\left(\frac{S_n^\rho}{S_n^*}\right) > \epsilon, \text{ i.o.}\right\} = 0.$$

Since $\epsilon$ was arbitrary, it follows that for all $\rho \in \Pi$

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\frac{S_n^\rho}{S_n^*}\right) \le 0, \qquad \text{with probability } 1. \qquad \square$$

An argument similar to that presented in [5] proves that $M_n = (S_n/S_n^*)$ is a nonnegative supermartingale that converges to a nonnegative random variable $Y$ such that $\boldsymbol{E}Y \le 1$.

Using the results in [5] one can easily extend the results to stationary ergodic horse race markets. We state the result and refer the reader to [5] to reconstruct the proof.

*Theorem 4:* Let $\{X_n : -\infty < n < \infty\}$ be the sequence of price relative vectors of a stationary ergodic horse race market with proportional transaction costs $\{\lambda_i : 1 \le i \le m\}$. Define the conditionally log-optimum policy $\rho^* = (b_1^*, b_2^*, \ldots)$ as follows:

$$b_n^* = \operatorname*{argmax}_{b\in\mathcal{B}} \boldsymbol{E}\left[\log(w(b, z_n)b^t X_n)\,\middle|\,X_1^{n-1}\right]$$

i.e., the policy invests all the wealth in the conditionally log-optimum portfolio given the past information. Then this policy $\rho^*$ is growth-optimal and the associated maximum growth rate $g^*$ is given by

$$g^* = \boldsymbol{E}\left[\log o(X_0)\,\middle|\,X_{-\infty}^{-1}\right] - H\left(X_0\,\middle|\,X_{-\infty}^{-1}\right) - \boldsymbol{E}\left[\log\left(\frac{1-\lambda(X_0)}{1+\lambda(X_0)}\right)(1-\delta(X_0, X_{-1}))\middle| X_{-\infty}^{-1}\right] \tag{20}$$

where $o(X_0) = o_i, \lambda(X_0) = \lambda_i$ if $X_0 = o_i e_i$, and $\delta(X_0, X_{-1})$ is the discrete $\delta$ function, i.e.,

$$\delta(X_0, X_{-1}) = \begin{cases} 1, & X_0 = X_{-1} \\ 0, & \text{otherwise.}\end{cases}$$

The next two results investigate the effect of the information structure on the achievable growth rate. Up to this point, we have assumed that the investor has perfect knowledge of the distribution of the market sequence $\{X_n : n \ge 0\}$. Instead, if the investor were to have only an estimate of the distribution, the achievable growth rate would necessarily be reduced. We show that this loss in growth rate is given by the relative entropy distance of the estimate from the true distribution.

*Theorem 5:* Let the sequence of price relatives $\{X_n : n \ge 1\}$ of a horse market with proportional transaction costs $\{\lambda_i : 1 \le i \le m\}$ be i.i.d. according to

$$\boldsymbol{P}(X_n = o_i e_i) = p_i, \qquad i = 1, \ldots, m.$$

Suppose $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_m)$ is the investor's estimate of the distribution of the asset price relatives $X_n$. Then the growth rate $\hat{g}$ of the conditionally log-optimum policy corresponding to the estimate $\hat{p}$ is given by

$$\hat{g} = g - D(p\|\hat{p}) \tag{21}$$

where $g$ is the growth rate corresponding to the true distribution $p$ and

$$D(p\|\hat{p}) = \sum_{j=1}^{m} p_j \log(p_j/\hat{p}_j)$$

is the relative entropy distance between the true distribution $p$ and the estimate $\hat{p}$.

*Remark 1:* Note that the loss in growth rate due to actions based on an incorrect distribution $\hat{p}$ does not depend on the transaction costs. This result does not extend to general markets.

*Remark 2:* Note that $D(p\|\hat{p}) \ge 0$, with equality if and only if $p = \hat{p}$. Thus $\hat{g} \le g$.

*Proof:* Since the investor estimates the true distribution to be $\hat{p}$, if follows form (10) that the optimal portfolio when $z_n = e_i$ is given by

$$\hat{b}_i(j) = \begin{cases} (\hat{p}_i) \Big/ \left(\hat{p}_i + \sum_{l\neq i}\left(\frac{1-\lambda_i}{1+\lambda_l}\right)\hat{p}_l\right), & j = i \\ \left(\left(\frac{1-\lambda_i}{1+\lambda_j}\right)\hat{p}_j\right) \Big/ \left(\hat{p}_i + \sum_{l\neq i}\left(\frac{1-\lambda_i}{1+\lambda_l}\right)\hat{p}_l\right), & j \neq i.\end{cases}$$

Therefore, the corresponding growth rate is given by

$$\begin{aligned} g(\hat{p}) &= \boldsymbol{E}\log\left(w(\hat{b}_n, z_n)\hat{b}_n^t X_n\right) \\ &= \sum_{i=1}^m p_i\left[\sum_{j\neq i} p_j \log\left(\frac{1-\lambda_i}{1+\lambda_j}\right) + \sum_{j=1}^m p_j \log(\hat{p}(j) o_j)\right] \\ &= \sum_{i=1}^m p_i(1-p_i)\log\left(\frac{1-\lambda_i}{1+\lambda_i}\right) + \sum_{i=1}^m p_i \log(p_i o_i) \\ &\quad - \sum_{i=1}^m p_i \log\left(\frac{p_i}{\hat{p}_i}\right) \\ &= g - D(p\|\hat{p}), \end{aligned}$$

where $D(p\|\hat{p})$ is the relative entropy distance between the true distribution $p$ and the estimate $\hat{p}$. $\square$

Thus an error in estimating the distribution results in an lower growth rate of wealth. Or equivalently, an error in the portfolio choice $b_n$ is penalized by a decrease in the growth rate of wealth. This property is exploited in the next section to show the existence of a universal policy for horse race markets with proportional transaction costs.

The next result characterizes the achievable growth rate when the investor has access to side information $\{Y_k : k \ge 1\}$. We assume that the sequence $\{(X_k, Y_k) : k \ge 1\}$ is i.i.d., and that the portfolio at time $k$ is allowed to depend on the observed value of $Y_k$.

*Theorem 6:* Let $\{Y_k : k \ge 1\}$ be side information for a horse race market with $\{(X_k, Y_k) : k \ge 1\}$ i.i.d., then the achievable growth rate $g_Y$ with side information is given by

$$g_Y = g + I(X;Y)$$

where $g$ is the growth rate without side information and $I(X;Y)$ is the mutual information between $X$ and $Y$.

*Proof:* The result follows from Theorem 5 and the fact that $I(X;Y) = D(p(x,y)\|p(x)p(y))$. $\square$

## III. Universal Investment in Horse Races

In the last section, the market was stochastic and the asset returns were distributed according to a known distribution. In this section, we remove all stochastic assumptions on the asset returns. This will yield a so-called individual sequence result.

As before, however, one of the assets pays off and all the other assets pay nothing. To emphasize the fact that the market price relatives are no longer stochastic we represent them by the sequence $\{x_n : n \geq 0\}$, with each $x_n \in \{o_i e_i : 1 \leq i \leq m\}$. As before, the sequence $\{z_n : n \geq 1\}$ will refer to the market opening portfolios of the investors, i.e.,

$$z_n = \frac{x_{n-1}}{\sum_{j=1}^{m} x_{n-1}(j)} \in \{e_j : 1 \leq j \leq m\}.$$

We make a simplifying assumption that the investor begins with all the wealth in asset 1, i.e., $z_1(1) = 1$. This is not a loss in generality since the wealth increases exponentially and the initial conditions wash out in the limit.

In this section, we restrict the investor to use stationary, first-order Markov self-financing policies, i.e., the decision at time $n$ is independent of the time instant $n$ and depends only on the previous market realization $z_n$. Therefore, any admissible investment policy $\rho$ can be equivalently described by a collection of $m$ vectors $(b_1^\rho, \ldots, b_m^\rho), b_i^\rho \in \mathcal{B}$, with the interpretation that at time $n$ the investor rebalances to the portfolio $b_i^\rho$ if $z_n = e_i$. Since each of the portfolios $b_i^\rho \in \mathcal{B}$, the set of all admissible policies is $\mathcal{B}^m$.

We do not restrict the investors to use nonanticipating policies. In fact, we allow the investors to choose the investment policy with hindsight, i.e., the stationary investment policy up until the $n$th market outcome can be chosen after observing the market sequence $x_1^n = \{x_k : 1 \leq k \leq n\}$. The main result of this section establishes that there is a universal nonanticipating policy that performs as well as any stationary policy, to first order in the exponent, even those chosen with hindsight. This universal policy, although Markov, is not stationary.

We introduce some new notation. We will denote the wealth factor $w(b_k, z_k)$ associated with the policy $\rho$ by $w_k^\rho$, i.e.,

$$w_k^\rho = w\left(b_i^\rho, e_i\right), \qquad \text{if } z_k = e_i. \tag{22}$$

The one-step wealth generated by policy $\rho$ at time $k$ on the market sequence $x_1^n$ is denoted by $W_k^\rho$, i.e.,

$$W_k^\rho = b_i^\rho(j) o_j, \qquad \text{if } z_k = e_i, z_{k+1} = e_j. \tag{23}$$

The wealth $S_n^\rho$ generated by the policy $\rho$ over the market sequence $x_1^n = \{x_1, \ldots, x_n\}, x_i \in \{o_1 e_1, \ldots, o_m e_m\}$, is given by

$$\begin{aligned} S_n^\rho &= \prod_{k=1}^{n} w_k^\rho W_k^\rho \\ &= \left[\prod_{k=1}^{m} \left(w(b_k^\rho, e_k) o_k\right)^{n_k}\right] \left[\prod_{i,j=1}^{m} \left(b_i(j)\right)^{n_{ij}}\right] \end{aligned} \tag{24}$$

where $n_{ij}$ is the number of times the pair $(e_i, e_j)$ occurs in $x_1^n$, and $n_i = \sum_j n_{ij}$. Collecting terms we have

$$\frac{1}{n} \log\left(S_n^\rho\right) = \sum_{i,j=1}^{m} p_n(i,j) \log\left(w(b_i^\rho, e_i) b_i(j) o_j\right) \tag{25}$$

where $p_n(i,j) = n_{ij}/n$ is the empirical distribution of the pair $(i,j)$. Let $\rho_n^*$ be the policy that maximizes the value of $S_n^\rho$, i.e., the best stationary policy with hindsight for a given realization $x_1^n$. We will denote the corresponding wealth by $S_n^*$, i.e., $S_n^* = \max_{\rho \in \mathcal{B}^m} S_n^\rho$ is the maximum wealth achievable on the sequence $x_1^n = (x_1, \ldots, x_n)$ given hindsight.

Next, we define our candidate universal policy $\hat{\rho}$. The policy $\hat{\rho}$ invests $d\rho / \int_{\mathcal{B}^m} d\rho$ dollars in the policy $\rho$ associated with $(b_1, \ldots, b_m) \in \mathcal{B}^m$, and then manages each of the little pools separately. At time $k$, a dollar invested in policy $\rho$ is worth $S_k^\rho$ and is invested in portfolio $z_k$. Since the initial wealth invested in the policy $\rho$ was $d\rho / \int_{\mathcal{B}^m} d\rho$, we have that the wealth $\hat{S}_k$ generated by the policy $\hat{\rho}$ is given by

$$\hat{S}_k = \frac{\int_{\mathcal{B}^m} S_k^\rho \, d\rho}{\int_{\mathcal{B}^m} d\rho}. \tag{26}$$

Furthermore, at time $k$, the policy $\rho$ dictates a move from the portfolio $z_k$ to the portfolio $b_k^\rho$, resulting in wealth $w_k^\rho S_k^\rho$ at $b_k^\rho$. The portfolio $\hat{b}_k$ corresponding to the universal policy $\hat{\rho}$ is obtained by weighting the portfolio of each of the individual policies $\rho$ by the corresponding wealth $w_k^\rho S_k^\rho$, i.e.,

$$\hat{b}_k = \frac{\int_{\mathcal{B}^m} w^\rho S_k^\rho b_k^\rho \, d\rho}{\int_{\mathcal{B}^m} w_k^\rho S_k^\rho \, d\rho}. \tag{27}$$

This policy $\hat{\rho}$ is the counterpart for the market with transaction costs to the universal policy defined in [7]. The policy $\hat{\rho}$ is nonanticipating and self-financing.

The following theorem establishes the main result of this section.

*Theorem 7:* Let $\rho_n^*$ be the best horizon-$n$ policy in hindsight and let $S_n^*$ be the wealth associated with $\rho_n^*$. Let $\hat{\rho}$ be the universal policy and $\hat{S}_n$ be the corresponding wealth. Then for all $n > n_{\min}$ and every market sequence $\{x_k : k \geq 1\}$

$$\frac{\hat{S}_n}{S_n^*} \geq \frac{e^{-(1+n_{\min})}}{n^{-m(m-1)}}$$

where

$$n_{\min} = 2\lambda_{\max}/(1 - \lambda_{\max})$$

and

$$\lambda_{\max} = \max\{\lambda_i : 1 \leq i \leq m\}.$$

*Remark 3:* The wealth $\hat{S}_n$ corresponding to the universal policy $\hat{\rho}$ tracks $S_n^*$, the maximum wealth achievable given hindsight, to within a polynomial factor. Thus the difference in growth rate $(1/n)\log(\hat{S}_n/S_n^*) \to 0$.

*Remark 4:* If $\lambda_{\max} < \frac{1}{3}$, then $n_{\min} < 1$. Therefore, the result holds for all $n \geq 1$.

*Proof:* Fix the time horizon $n$ and denote the best stationary policy with hindsight by $\rho^*$. In the rest of the proof we will denote $W_n^{\rho^*}, b_n^{\rho^*}$ and $S_n^{\rho^*}$ by $W_n^*, \rho_n^*$ and $S_n^*$, respectively.

Fix $\alpha \in (0,1)$ and a policy $\rho_v$ with the corresponding vector $(v_1, v_2, \ldots, v_m) \in \mathcal{B}^m$. Define the policy $\rho_\alpha = (1-\alpha)\rho^* + \alpha\rho_v$, i.e., the policy $\rho_\alpha$ at time $k$ rebalances to $b_i = (1-\alpha)b_i^* + \alpha v_i$ if $z_k = e_i$. As in the case with policy $\rho^*$, $W_n^\alpha, b_n^\alpha$, and $S_n^\alpha$ will denote $W_n^{\rho_\alpha}, b_n^{\rho_\alpha}$, and $S_n^{\rho_\alpha}$, respectively.

The wealth $W_k^\alpha$ generated by the policy $\rho_\alpha$ at time $k$

$$\begin{aligned} W_k^\alpha &= (1-\alpha)W_k^* + \alpha W_k^{\rho_v}, \\ &\geq (1-\alpha)W_k^*. \end{aligned} \tag{28}$$

One feasible way of correcting the portfolio from $z_k$ to $b_k^\alpha$ is to first move from $z_k$ to $b_k^*$ and then move a proportion $\gamma$ from $b_k^*$ to $b_k^{\rho_v}$ such that the final portfolio is $b_k^\alpha$. Let $w = w(b_k^{\rho_v}, b_k^*)$. Then $\gamma$ is the solution of the equation

$$\alpha = \frac{\gamma w}{\gamma w + (1-\gamma)}.$$

One dollar at $z_k$ nets $((1-\gamma) + \gamma w)w_k^*$ dollars at $b_k^\alpha$. Substituting for $\gamma$, we get

$$\begin{aligned} (1-\gamma) + w\gamma &= \left(1 + \alpha(1/w - 1)\right)^{-1} \\ &\geq 1 - \alpha n_{\min}. \end{aligned}$$

It follows that the repositioning factor $w_n^\alpha$ can be bounded by

$$w_n^\alpha \geq (1-\alpha n_{\min})\, w_k^*. \tag{29}$$

The total wealth $S_n^\alpha$ generated by the policy $\rho^\alpha = (1-\alpha)\rho^* + \alpha\rho_v$ over the market sequence $x_1^n$ is given by

$$\begin{aligned} S_n^\alpha &= \prod_{k=1}^n w_n^\alpha W_n^\alpha \\ &\geq \prod_{k=1}^n ((1-\alpha n_{\min})w_k^*)\,((1-\alpha)W_k^*) \qquad (30) \\ &= (1-\alpha n_{\min})^n (1-\alpha)^n S_n^* \qquad (31) \end{aligned}$$

where (30) follows from the lower bounds developed in (28) and (29). Define a set $G$ as follows:

$$G = \{\rho \mid \exists (v_1, \dots, v_m) \in \mathcal{B}^m \text{ with } (b_1^\rho, \dots, b_m^\rho) = (1-\alpha)(b_1^*, \dots, b_m^*) + \alpha(v_1, \dots, v_m)\}.$$

From (31) we have that for all $\rho \in G$

$$S_n^\rho \geq (1-\alpha)^n (1-\alpha n_{\min})^n S_n^*.$$

Using the characterization in (26) and the bounds developed above, the wealth $\hat{S}_n$ generated by the universal policy $\hat{\rho}$ can be bounded by

$$\begin{aligned} \hat{S}_n &= \frac{\int_{\mathcal{B}^m} S_n^\rho \, d\rho}{\int_{\mathcal{B}^m} d\rho} \\ &\geq \frac{\int_G S_n^\rho \, d\rho}{\int_{\mathcal{B}^m} d\rho} \\ &\geq \frac{\int_G (1-\alpha n_{\min})^n (1-\alpha)^n S_n^* \, d\rho}{\int_{\mathcal{B}^m} d\rho} \\ &= (1-\alpha)^n (1-\alpha n_{\min})^n S_n^* \cdot \frac{\int_G d\rho}{\int_{\mathcal{B}^m} d\rho} \\ &= (1-\alpha)^n (1-\alpha n_{\min})^n S_n^* \alpha^{m(m-1)}. \end{aligned}$$

Choose $\alpha = 1/n$. Then, for all $n > n_{\min}$, we have the following lower bound on $\hat{S}_n/S_n^*$:

$$\frac{\hat{S}_n}{S_n^*} \geq \left(1 - \frac{1}{n}\right)^n \left(1 - \frac{n_{\min}}{n}\right)^n \left(\frac{1}{n}\right)^{-m(m-1)} \geq \frac{e^{-(1+n_{\min})}}{n^{m(m-1)}}.$$

This proves the theorem. □

Although not immediately obvious from the proof, we have used Laplace's method of integration to get a lower bound on the integral characterizing the wealth generated by the universal policy.

The following corollary establishes that the universal policy $\hat{\rho}$ achieves the same growth rate as the best stationary Markov policy chosen in hindsight.

*Corollary 1:* The wealth sequence $\{\hat{S}_n : n \geq 1\}$ generated by the universal policy $\hat{\rho}$ asymptotically does as well as the sequence $\{S_n^* : n \geq 1\}$ generated by the policies chosen in hindsight, in the sense that for every market sequence $\{x_n : n \geq 1\}$

$$\liminf_{n\to\infty} \frac{1}{n} \log \left(\frac{\hat{S}_n}{S_n^*}\right) \geq 0. \tag{32}$$

*Proof:* The corollary follows immediately from Theorem 7. □

Since the universal policy does as well as the best policy in hindsight, it immediately follows that if the market is stochastic then the universal policy achieves the optimal growth rate corresponding to the true market distribution.

*Corollary 2:* In a horse race market with i.i.d. price relative vectors and maximum achievable growth rate $g$, the wealth sequence $\{\hat{S}_n : n \geq 1\}$ associated with the universal policy $\hat{\rho}$ satisfies

$$\liminf_{n\to\infty} \frac{1}{n} \log(\hat{S}_n) = g, \qquad \text{with probability 1.} \tag{33}$$

*Remark 5:* This result remains true even if the market sequence $\{X_n : n \geq 1\}$ is first-order Markov instead of i.i.d. The universal policy $\hat{\rho}$ achieves the maximal growth rate corresponding to the best Markov policy.

*Proof:* Let $\rho_p$ be a conditionally log-optimum policy for the stochastic market and $\{S_n^p : n \geq 1\}$ be the corresponding wealth stream. Since the wealth $S_n^*$ associated with the horizon-$n$ optimal policy $\rho_n^*$ is always no less than $S_n^p$ for all time instants $n$, it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log S_n^* \geq \limsup_{n\to\infty} \frac{1}{n} \log S_n^p. \tag{34}$$

From Corollary 1 it follows that for all market sequences

$$\liminf_{n\to\infty} \frac{1}{n} \log \hat{S}_n \geq \limsup_{n\to\infty} \frac{1}{n} \log S_n^p. \tag{35}$$

Also, from Theorem 3 it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \log S_n^p \geq \limsup_{n\to\infty} \frac{1}{n} \log \hat{S}_n, \qquad \text{with probability 1.} \tag{36}$$

Therefore, from (35) and (36) it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log \hat{S}_n = \lim_{n\to\infty} \frac{1}{n} \log S_n^p, \qquad \text{with probability 1.}$$

Theorem 2 identifies the limit as $g$; therefore,

$$\lim_{n\to\infty} \frac{1}{n} \log \hat{S}_n = g, \qquad \text{with probability 1.} \tag{37}$$

□

Thus the universal policy is able to "learn" the optimal policy corresponding to the unknown i.i.d. measure on the horse race market "on the fly" and still achieve the same growth rate.

## IV. Conclusion

Growth optimal investment in horse race or erodible asset markets with proportional transactions costs is similar to the problem without costs. For the stochastic horse race markets we show that the conditionally log-optimum policy still remains growth-optimal. At time $n$, if $z_n = e_i$, the optimal policy invests all the wealth in the portfolio $b_n^\lambda$ given by

$$b_n^\lambda = \operatorname*{argmax}_{b\in\mathcal{B}} \boldsymbol{E}\left[\log(w(b, e_i) b^t X_n) \,\middle|\, X_1^{n-1}\right].$$

The portfolio $b_n^\lambda$ is explicitly given as follows:

$$b_n^\lambda(j) = \begin{cases} \dfrac{p_i^{(n-1)}}{p_i^{(n-1)} + \sum_{l\neq i} \left(\frac{1-\lambda_i}{1+\lambda_l}\right) p_l^{(n-1)}}, & j = i, \\[2ex] \dfrac{\left(\frac{1-\lambda_i}{1+\lambda_j}\right) p_j^{(n-1)}}{p_i^{(n-1)} + \sum_{l\neq i} \left(\frac{1-\lambda_i}{1+\lambda_l}\right) p_l^{(n-1)}}, & j \neq i, \end{cases} \qquad j = 1, \dots, m$$

where $p_j^{(n-1)} = \boldsymbol{P}(X_n = o_j e_j | X_0^{n-1})$. This policy is very similar to the optimal policy $\{b_n^0 : n \geq 1\}$ given by Algoet [5] for the frictionless case, where

$$\begin{aligned} b_n^0 &= \operatorname*{argmax}_{b\in\mathcal{B}} \boldsymbol{E}\left[\log\left(b^t X_n\right) \,\middle|\, X_1^{n-1}\right] \\ &= p^{(n-1)}. \end{aligned}$$

The optimal policy in a horse race market with transactions costs has the following simple characterization. If asset $i$ wins, keep a proportion $p_i^{(n-1)}$ of it and sell $(1 - p_i^{(n-1)})$. Take the proceeds $(1-\lambda_i)(1 - p_i^{(n-1)})$ of the sale and invest it in the remaining assets in the proportions

$$(p_1^{(n-1)}, p_2^{(n-1)}, \dots, p_{i-1}^{(n-1)}, p_{i+1}^{(n-1)}, \dots, p_m^{(n-1)})$$

i.e., invest $(1-\lambda_i)p_j^{(n-1)}$ in asset $j \neq i$. As a result, the final wealth is distributed according to

$$(\gamma_1 p_1^{(n-1)}, \ldots, \gamma_{i-1} p_{i-1}^{(n-1)}, p_i^{(n-1)}, \gamma_{i+1} p_{i+1}^{(n-1)}, \ldots, \gamma_m p_m^{(n-1)})$$

where $\gamma_j = (1-\lambda_i)/(1+\lambda_j)$. Thus the intent is always the same—place sell and buy orders as if there were no transactions costs ([5], [3]) and accept whatever you get as a result.

The maximum achievable growth rate $g_\lambda$ in a stationary, ergodic market with transaction costs is given by

$$\begin{aligned} g_\lambda &= \max_{b \in \mathcal{B}} \boldsymbol{E}\left[\log(w(b, z_0) b^t X_1) \,\middle|\, X_{-\infty}^0\right] \\ &= \boldsymbol{E}\left[\log o(X_0) \,\middle|\, X_{-\infty}^{-1}\right] - H\left(X_0 \,\middle|\, X_{-\infty}^{-1}\right) \\ &\quad - \boldsymbol{E}\left[\log\left(\frac{1-\lambda(X_0)}{1+\lambda(X_0)}\right)(1-\delta(X_0, X_{-1})) \,\middle|\, X_{-\infty}^{-1}\right] \end{aligned}$$

which again bears a very close similarity to the maximum growth rate in frictionless markets $g_0$ given by

$$\begin{aligned} g_0 &= \max_{b \in \mathcal{B}} \boldsymbol{E}\left[\log(b^t X_1) \,\middle|\, X_{-\infty}^0\right] \\ &= \boldsymbol{E}\left[\log o(X_0)\right] - H\left(X_0 \,\middle|\, X_{-\infty}^{-1}\right). \end{aligned}$$

The similarity also manifests itself in the dual relationship of the maximum growth rates and minimum information rates.

We also show that the universal investment results in [7] can be easily extended to the case of markets with proportional transaction costs. The cost of universality is only polynomial in the time horizon and does not affect the asymptotic growth rate.

Horse race markets are a very special extreme case of general markets but may provide an example of the general behavior of optimal strategies in general markets. For example, in the stochastic framework, Barron and Cover [17] show that the increase in growth due to side information is maximized for horse race markets. Cover and Ordentlich [2], [8] set up a minimax game where the investor chooses the stationary investment policy and nature chooses the sequence of market price relative vectors. They show that horse race markets achieve the minimax equilibrium for this game.

Horse race markets are considerably simpler to analyze because the sequence of market opening portfolios of the investors is the same, independent of the policy. This is not the case in general markets; as a result, comparing policies is hard. We have a solution of the problem when the market price relatives are i.i.d. or finite-order Markov. The solution relies on results from Markov decision problems and some selection theorems [15]. We believe that the solution method presented in this work can be extended to the general case by defining a suitable coupling between the wealth processes of the admissible policies.

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## REFERENCES

[1] R. Bell and T. M. Cover, "Game-theoretic optimal portfolios," *Manag. Sci.*, vol. 34, no. 6, pp. 724–733, 1988.
[2] T. M. Cover and E. Ordentlich, "Universal portfolios with side information," *IEEE Trans. Inform. Theory*, vol. 42, pp. 348–363, Mar. 1996.
[3] J. Kelly, "A new interpretation of information rate," *Bell Syst. Tech. J.*, vol. 35, pp. 917–926, 1956.
[4] L. Breiman, "Optimal gambling systems for favorable games," in *Proc. 4th Berkeley Symp. Math. Statist. Probab.*. Berkeley, CA: Univ. California Press., 1961, vol. 1.
[5] P. H. Algoet and T. M. Cover, "Asymptotic optimality and asymptotic equipartition properties of log-optimum investment," *Ann. Probab.*, vol. 16, no. 2, pp. 876–898, 1988.
[6] P. Algoet, "On the convex duality between maximum growth rate and minimum information rates," unpublished, 1997.
[7] T. M. Cover, "Universal portfolios," *Math. Finance*, vol. 1, no. 1, pp. 1–29, 1991.
[8] E. Ordentlich and T. M. Cover, "The cost of achieving the best portfolio in hindsight," *Math. Oper. Res.*, vol. 23, no. 4, pp. 960–982, 1998.
[9] M. H. A. Davis and A. R. Norman, "Portfolio selection with transaction costs," *Math. Oper. Res.*, vol. 15, no. 4, pp. 676–713, 1990.
[10] A. Kalai and A. Blum, "Universal portfolios with and without transaction costs," in *Proc. 10th Annu. Conf. Learning Theory*, 1997, pp. 309–313.
[11] M. I. Taksar, M. J. Klass, and D. Assaf, "Diffusion model for optimal portfolio selection in the presence of brokerage fees," *Math. Oper. Res.*, vol. 13, no. 2, pp. 277–294, 1988.
[12] M. Akian, J. L. Menaldi, and A. Sulem, "On an investment-consumption model with transaction costs," *SIAM J. Contr. Optim.*, vol. 34, no. 1, pp. 329–364, 1996.
[13] G. N. Iyengar, "Analysis of growth rate in the presence of transactions costs—Continuous time," Dept. Statist., Stanford Univ., Tech. Rep. 98, 1997.
[14] M. Akian, A. Sulem, and M. I. Taksar, Dynamic optimization of long term growth rate for a mixed portfolio with transactions costs, , preprint.
[15] G. N. Iyengar, "Growth optimal investment in discrete time markets with costs," *Math. Oper. Res.*, 2000, submitted for publication.
[16] S. E. Shreve and H. M. Soner, "Optimal investment and consumption with transaction costs," *Ann. Appl. Probab.*, vol. 4, no. 3, pp. 609–692, 1994.
[17] A. R. Barron and T. M. Cover, "A bound on the financial value of information," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1097–1100, Sept. 1988.

# Iterated Logarithmic Expansions of the Pathwise Code Lengths for Exponential Families

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***Abstract*—Rissanen's minimum description length (MDL) principle is a statistical modeling principle motivated by coding theory. For exponential families we obtain pathwise expansions, to the constant order, of the predictive and mixture code lengths used in MDL. The results are useful for understanding different MDL forms.**

***Index Terms*—Exponential family, law of iterated logarithm, minimum description length (MDL), mixture code, predictive code.**

## I. INTRODUCTION AND BACKGROUND

The minimum description length (MDL) principle was introduced by Rissanen as a fundamental principle to model data, see [15], [17], and the reference list in [18]. If we encode data from a source by prefix

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