

The relation of description rate and investment growth rate

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Abstract — We have shown that if one invests in the outcome of a random variable X , where investment consists of gambling at any odds, then every bit of description of X increases the doubling rate by one bit. However, if the provider of the information has access only to V , a random variable jointly distributed with X , then this maximal efficiency is not generally possible. We find the increase $\Delta(R)$ in doubling rate for a description of V at rate R for the jointly Gaussian and jointly binary cases. We investigate the extension to multivariate Gaussian random variables. We prove a general result for the derivative of $\Delta(R)$ at $R = 0$.

We then consider the problem in which there are k separate encoders and each observes a random variable V_i correlated with X . We find how efficiently these encoders, without cooperation, help the investor who is interested in X .

SUMMARY

Suppose one gambles on the outcome of a random variable X . The investor distributes his wealth according to $b(x)$ and the investment pays odds of $o(x)$ for one. Also suppose that the description of another random variable V , which has a known joint distribution with X , at the rate of R bits is allowed. Let $\Delta(R)$ be the maximum increase in the doubling rate from no description to a description of rate R . It can be seen that $\Delta(R)$ is a concave and nondecreasing function of R . We can show [2] that

$$\Delta(R) = \max_{p(\tilde{v}|v,x): I(V;\tilde{V}) \leq R, \tilde{V} \rightarrow V \rightarrow X} I(\tilde{V}; X).$$

We define *initial efficiency* as the derivative of $\Delta(R)$ at the origin. Initial efficiency is the maximum possible increase in $\Delta(R)$ per bit of description. For $V = X$, $\Delta(R) = R$; hence the efficiency is 1. However, for a general V , the efficiency is generally less than 1. We find $\Delta(R)$ and examine the efficiency of the jointly binary and Gaussian cases.

Theorem 1 *Suppose V and X are both Bernoulli($\frac{1}{2}$) random variables associated by a binary symmetric channel with crossover probability p . The $\Delta(R)$ curve is given by*

$$(R, \Delta(R)) = (1 - h(\alpha), 1 - h(\alpha * p))$$

where $0 \leq \alpha \leq 1$, h is the binary entropy function and $*$ is the cascade operation.

We use a lemma by Wyner and Ziv, known as ‘Mrs. Gerber’s Lemma’ [4] to prove the optimality of the descriptions in the above theorem. The initial efficiency can be calculated as $(1 - 2p)^2$.

Theorem 2 *Suppose X and V are jointly Gaussian with correlation ρ . Then*

$$\Delta(R) = \frac{1}{2} \log\left(\frac{1}{1 - \rho^2(1 - 2^{-2R})}\right).$$

The proof of optimality in the Gaussian problem requires a lemma by Bergmans, which is a conditional version of the entropy power inequality [1]. We note that the initial efficiency is ρ^2 .

A natural generalization of this theorem is to multivariate Gaussian. Suppose $V^n \sim N(0, K_V)$, $Z^n \sim N(0, K_Z)$, V^n and Z^n are independent and $X^n = V^n + Z^n$. By changing the coordinate system, we can obtain diagonal covariance matrices and hence transform the problem to one on parallel subchannels with a total rate constraint. The solution is given by water-filling in the entropy domain. We distribute the total rate so that the derivative of $\Delta(R)$ with respect to R at the operating point is the same for all the subchannels used.

We note that in all the problems examined, the initial efficiency is related to the correlation between V and X . We define the *maximal correlation* between V and X as the supremum of $Ef(X)g(V)$, where the supremum is over all functions f and g such that $Ef(X) = Eg(V) = 0$ and $Ef^2(X) = Eg^2(V) = 1$. Maximal correlation depends only on the joint distribution of V and X and is independent of the actual labeling. Conditions under which the maximal correlation can be attained have been investigated by Rényi [3]. Our next theorem examines the relationship between the initial efficiency and maximal correlation.

Theorem 3 *Initial efficiency is equal to the square of the maximal correlation between V and X .*

Next we consider k separate senders. We are interested in the increase in the doubling rate, Δ , for gambling on X when sender i observes V_i correlated with X and the senders operate at respective rates R_1, \dots, R_k . We prove an achievable region for $(R_1, \dots, R_k, \Delta)$, and show that a Slepian-Wolf type of rate region is achievable for this investment problem.

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