The Entropy of Markov Trajectories

Laura Ekroot
Department of Electrical Engineering

Thomas M. Cover
Departments of Electrical Engineering and Statistics

Stanford University
Information Systems Lab
Stanford, CA 94305

Abstract

We find the entropy of trajectories from state i to state j for finite state irreducible aperiodic Markov chains.

Summary

Consider a finite state irreducible Markov chain with transition matrix P and initial state $X_1 = i$. The entropy rate

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{H(X_1, X_2, \ldots, X_n)}{n}$$

is given by

$$H(\mathcal{X}) = -\sum_{i,j} \mu_i P_{ij} \log P_{ij}$$

where μ is the stationary distribution which solves the equations $\mu_j = \sum_i \mu_i P_{ij}$, for all j.

Definition 1 *A* trajectory from state i to state j *of a Markov chain is a path with initial state i, final state j, and no intervening state equal to j.*

The probability $p(t_{ij})$ of a trajectory $t_{ij} = ix_2x_3\ldots x_kj$ is $p(t_{ij}) = P_{ix_2}P_{x_2x_3}\cdots P_{x_kj}$. This, of course, is the conditional probability of the trajectory t_{ij} given $X_1 = i$.

Since a trajectory from i to j terminates the first time it reaches j, no trajectory t_{ij} is a prefix of any other trajectory from i to j. This, together with the irreducibility of P, implies that $\sum_{t_{ij}\in\mathcal{T}_{ij}} p(t_{ij}) = 1$, where $\mathcal{T}_{ij}$ is the set of all trajectories from i to j. Thus, given initial state $X_1 = i$, the *random trajectory T_{ij}* is a finite length sequence drawn according to the probability mass function $p(t_{ij})$.

Definition 2 *The* entropy H_{ij} of the trajectory from i to j *is*

$$H_{ij} = H(T_{ij}) = -\sum_{t_{ij}\in\mathcal{T}_{ij}} p(t_{ij}) \log p(t_{ij}).$$

Our objective will be to determine a closed form expression for H_{ij}. We begin by developing a recurrence relation for H_{ij}. From this we can determine H_{ii}, the entropy of the path that loops back to the intial state. Subsequently, this solution for H_{ii} will be substituted into the recurrence to complete the determination of H_{ij}, for all i, j.

Let $P_{i\cdot}$ denote the i^{th} row of the Markov transition matrix P. Then

$$H(P_{i\cdot}) = -\sum_j P_{ij} \log P_{ij}$$

is the entropy of the first step of a trajectory originating in state i. The recurrence relation

$$H_{ij} = H(P_{i\cdot}) + \sum_{k\neq j} P_{ik} H_{kj}, \qquad (1)$$

follows from the chain rule for entropy. Specifically, the entropy of a random trajectory is given by the entropy of the first step plus the conditional entropy of the remaining trajectory given the first step.

Theorem 1 *The entropy H_{ii} of the random trajectory T_{ii} from i back to i is given by*

$$H_{ii} = \frac{H(\mathcal{X})}{\mu_i},$$

where μ_i is the stationary probability for state i.

The entropy of the random trajectory T_{ii} has an interpretation as the product of the expected number of steps $1/\mu_i$ to return to state i and the per-step entropy rate $H(\mathcal{X})$ of the stationary Markov chain.

We now complete the determination of H_{ij} for all i and j by substituting the diagonal terms H_{ii} derived in Theorem 1 into the recurrence relation (1) and solving for the off-diagonal terms.

Theorem 2 *If P is the transition matrix of an irreducible aperiodic finite state Markov chain, then the matrix H of trajectory entropies is given by*

$$H = K - \tilde{K} + H_\Delta,$$

where

$$K = (I - P + A)^{-1}(H^* - H_\Delta).$$

Here $\tilde{K}$ is a matrix in which the ij^{th} element $\tilde{K}_{ij}$ equals the diagonal element K_{jj} of K; $A_{ij} = \mu_j$ is the matrix of stationary probabilities; $H^*_{ij} = H(P_{i\cdot}) = -\sum_k P_{ik}\log P_{ik}$ is the matrix of single-step entropies; and H_Δ is the diagonal matrix with entries $(H_\Delta)_{ii} = H(\mathcal{X})/\mu_i$.

The proofs of Theorems 1 and 2 will appear in [1].

References

[1] L. Ekroot and T. M. Cover. "The entropy of Markov trajectories". Submitted to IEEE Transactions on Information Theory, 1991.

[2] J. G. Kemeny and J. L. Snell. *Finite Markov Chains*. D. Van Nostrand Company, Inc., 1960.