Universal portfolios with short sales and margin

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Abstract — **In the horizon-free universal portfolio [1, 2] and the fixed horizon minimax regret portfolio [3], the only allowed portfolios are those that allocate a nonnegative proportion of wealth to each of the assets. It was shown that a Dirichlet(1/2)-weighting of constant rebalanced portfolios was asymptotically minimax regret in growth rate and therefore universal. We extend these results to bounded stocks and portfolios allowing margin and short sales.**

I. Summary

Consider the stock vector $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im})$, where x_{ik} is the price-relative for stock k on day i, *i.e.*, the ratio of the closing price to the opening price of stock k on day i. Here we treat the bounded stock case in which $x_{ik}/x_{il} \leq r$, for $k, l = 1, 2, \ldots, m$. Thus, for such bounded stocks, the broker would demand only that the portfolio $\mathbf{b}$ satisfy $\mathbf{b}^t\mathbf{x} \geq 0$ for all $\mathbf{x} \in A = \{\mathbf{x} \in \mathbb{R}^m : x_i/x_j \leq r, \text{ for } i, j = 1, 2, \ldots, m\}$. Thus, allowed portfolios lie in the dual cone $B = A^+ = \{\mathbf{b} \in \mathbb{R}^m : \sum_{i=1}^m b_i = 1, \mathbf{b}^t\mathbf{x} \geq 0, \mathbf{x} \in A\}$. We find the extreme points of the dual cone B, interpret them as short sales and margin sales, and characterize all transactions as a convex combination of the extreme transactions.

We then examine the possibility of universal portfolios in this market, *i.e.*, nonanticipating portfolios (including short sales and margin) which do as well (to first order in the growth rate) against any specific stock sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ as the best constant rebalanced portfolio $\mathbf{b}^* \in B$, *i.e.*, the portfolio that maximizes $S_n = \prod \mathbf{b}^t\mathbf{x}_i$. We define, for each $\mathbf{b} \in B$,

$$S_n(\mathbf{b}) = \prod_{i=1}^n \mathbf{b}^t\mathbf{x}_i,$$

and

$$S_n^* = \max_{\mathbf{b} \in B} S_n(\mathbf{b}),$$

where S_n^* is the wealth associated with the best constant rebalanced portfolio in hindsight, given the actual stock sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. We define the universal portfolio

$$\hat{\mathbf{b}}_{i+1} = \frac{\int_B \mathbf{b} S_i(\mathbf{b})\mu(d\mathbf{b})}{\int_B S_i(\mathbf{b})\mu(d\mathbf{b})},$$

and show that for μ equal to the uniform measure on B the associated wealth,

$$\hat{S}_n = \prod_{i=1}^n \hat{\mathbf{b}}_i^t\mathbf{x}_i,$$

achieves

$$\frac{\hat{S}_n}{S_n^*} \geq \frac{c}{n^{m-1}},$$

for all market sequences $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Thus, this is an individual, or worst case, sequence result. We also show that setting μ equal to the Dirichlet$(1/2, \ldots, 1/2)$ measure on the components of a simplicial decomposition of B achieves

$$\frac{\hat{S}_n}{S_n^*} \geq \frac{c}{n^{(m-1)/2}},$$

again for all market sequences. In each of the above bounds, c depends on r and m, but not on n or $\mathbf{x}^n$.

Thus, the nonanticipating portfolio $\hat{\mathbf{b}}_i$, $i = 1, 2, \ldots, n$, tracks S_n^* to within a polynomial factor, thereby achieving the same asymptotic growth rate of wealth as the best constant rebalanced portfolio with hindsight.

We then apply this algorithm to actual market sequences over a 20 year period to determine the gains that can be achieved for bounded stocks, with short sales and margin. In some cases, the wealth factors are quite dramatic.

References

[1] T. Cover. *Universal portfolios. Mathematical Finance*, 1(1): 1-29, January 1991.

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[3] E. Ordentlich, T. Cover. *The cost of achieving the best portfolio in hindsight.* To appear *Mathematics of Operations Research*, 1998.