The Equivalence of Optimal Market Gain and Minimax Regret Universal Portfolios

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Abstract — **Suppose one is given a set of joint distributions $\{p_\theta(\mathbf{x})\}$ over stock market outcomes $\mathbf{x} \in \mathbb{R}^m$. We ask what actions θ should an investor take, and with what frequency, to maximize the investor's advantage over the market. In another question we ask for the universal portfolio minimizing the maximum regret in the growth rate of wealth. We argue, in parallel with Gallager's proof for the equivalence of channel capacity and minimum redundancy in data compression, that both problems have the same answer $\min_{\mathbf{b}} \max_\theta \int p_\theta \ln \frac{\mathbf{b}_\theta^t \mathbf{x}}{\mathbf{b}^t \mathbf{x}}$.**

Summary

Let $\{p_\theta\}$ be a family of probability densities on $\mathbf{x} \in \mathbb{R}^m$. Here $p_\theta(\mathbf{x})$ is the density of the market vector $\mathbf{x}$ if action θ is taken. The i^{th} component of the stock vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ is the price-relative of the i^{th} stock. A choice of portfolio $\mathbf{b} \in \mathbb{R}^m$, $\sum \mathbf{b}_i = 1$, $\mathbf{b}_i \geq 0$, results in a wealth-relative $\mathbf{b}^t\mathbf{x}$ and an associated growth rate of wealth $E \ln \mathbf{b}^t\mathbf{X}$.

We undertake a sort of signalling through the market in which the investor signals with action θ, resulting in distribution $p_\theta(\mathbf{x})$. The log optimal investment action $\mathbf{b}_\theta$ is the portfolio achieving $W^*(X|\theta) = \max_{\mathbf{b}} \int p_\theta(\mathbf{x}) \ln \mathbf{b}^t\mathbf{x}\, d\mathbf{x}$.

We envisage the sort of actions θ that might be available. For example, θ might be a choice to merge companies, a restriction of a company from competing with another, a deferral of dividends, or a purchase of small amounts of a given stock to prop up the price.

Suppose action θ is taken with probability $\pi(\theta)$. The resulting growth rate of wealth is

$$W^*(X|\theta) = \int W^*(X|\theta = \theta)\pi(\theta)d\theta.$$

The general market investor, uninformed of the action θ, can achieve maximum growth rate

$$W^*(X) = \max_{\mathbf{b}} \int\int \pi(\theta)p_\theta(\mathbf{x}) \ln \mathbf{b}^t\mathbf{x}\, d\mathbf{x}d\theta$$

using log optimal investment $\mathbf{b}$ against the marginal distribution

$$p(\mathbf{x}) = \int p_\theta(\mathbf{x})\pi(\theta)d\theta.$$

Thus the excess growth rate of wealth $\Delta(X, \theta)$ is given by

$$\Delta(X, \theta) = W^*(X|\theta) - W^*(X).$$

Let

$$\Delta^* = \max_{\pi(\theta)} \Delta(X, \theta)$$

be this maximum.

The result of the action θ on the market is not the transmission of information $I(\theta; X)$, but the generation of excess wealth at rate $\Delta^* = \max_\pi \Delta(\theta; X)$.

We now consider another problem. If the state of nature θ is known, one's optimal growth rate of wealth is $W^*(X|\theta)$. Suppose θ is unknown. What portfolio $\mathbf{b}$ should be used? The regret from using $\mathbf{b}$ instead of $\mathbf{b}_\theta$ is

$$E_\theta \ln \mathbf{b}_\theta^t X - E_\theta \ln \mathbf{b}^t X = \int p_\theta(\mathbf{x}) \ln \frac{\mathbf{b}_\theta^t\mathbf{x}}{\mathbf{b}^t\mathbf{x}} d\mathbf{x},$$

where $\mathbf{b}_\theta$ is log optimal with respect to $p_\theta(\mathbf{x})$. We now choose $\mathbf{b}$ to minimize the regret of the growth rate of wealth in the worst case. This minimax regret is

$$R^* = \min_{\mathbf{b}} \max_\theta \int p_\theta(\mathbf{x}) \ln \frac{\mathbf{b}_\theta^t\mathbf{x}}{\mathbf{b}^t\mathbf{x}} d\mathbf{x}.$$

The portfolio $\mathbf{b}^*$ achieving R^* is the universal portfolio with respect to $\{p_\theta\}$. The main theorem is as follows:

Theorem $\quad \Delta^* = R^*.$

Thus the minimax regret from the universal portfolio $\mathbf{b}^*$ is the same as the maximum increase in the growth rate of wealth induced by the optimal density $\pi^*(\theta)$ on the available actions θ. Moreover, $\mathbf{b}^* = \mathbf{b}_{p^*}$, where p^* is induced by π^*. The proof parallels Gallager's well known unpublished proof, via the fundamental theorem of game theory, that minimax redundancy in data compression is equal to channel capacity. We use the definition of portfolio distance $\Delta(p \parallel q) = \int p(\mathbf{x}) \ln \frac{\mathbf{b}_p^t\mathbf{x}}{\mathbf{b}_q^t\mathbf{x}}$ defined in Barron and Cover [2] (see also Cover and Thomas [3].)

A number of additional properties of $\Delta(p \parallel q)$ are derived in addition to the known inequality [2]

$$\Delta(p \parallel q) \leq \int p \ln \frac{p}{q}.$$

References

[1] Gallager, 1968, unpublished.

[2] Barron, A., and Cover, T., "A Bound on the Financial Value of Information," *IEEE Transactions of Information Theory*, 34(5): 1097-1100, September 1988.

[3] Cover, T., and Thomas, J., *Elements of Information Theory*, Wiley, 1991.

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