The Asymptotic Equivalence of Investing with and without Replacement

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Abstract and Summary

Consider the following scenarios for a sequence of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of price relatives corresponding to the history of a finite collection of stocks over a period of n investment periods: 1) Nothing is known about the sequence of vectors; 2) The vectors in the sequence are known, although the order is not known and the vectors are drawn independently with replacement from this set; 3) The collection of vectors is known, although the order is unknown, but the vectors are drawn without replacement from this set.

Clearly the amount of wealth that can be generated in these scenarios increases as the amount of information increases. For example, in scenario 2 one knows the empirical distribution of the market, whereas in 1, one does not. In 3, end-play can be used.

We shall argue, for bounded vector sequences, that the universal portfolio algorithm [1]

$$\hat{\mathbf{b}}_{k+1} = \frac{\int \mathbf{b} \prod_{i=1}^{k} \mathbf{b}^t \mathbf{x}_i db}{\int \prod_{i=1}^{k} \mathbf{b}^t \mathbf{x}_i db}$$

for scenario 1 will perform as well to first order in the exponent as the best algorithms in scenarios 2 and 3. Thus even end-play on a known collection of vectors of price relatives cannot outperform this universal portfolio based on no knowledge whatsoever, at least to first order in the exponent.

The growth rate of wealth in all three scenarios is given, to first order in the exponent, by the doubling rate W^* (a generalization of entropy rate), which is given by

$$W^* = \max_{\mathbf{b}} \frac{1}{n} \sum_{i=1}^{n} \log \mathbf{b}^t \mathbf{x}_i,$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is the sequence of vectors of price relatives for the n trading days, and the maximization is over all portfolios

$$\mathbf{b} = (b_1, b_2, \ldots, b_m), b_i \geq 0, \sum_{i=1}^{m} b_i = 1.$$

Thus the wealth

$$S_n = \prod_{i=1}^{n} \mathbf{b}_i^t \mathbf{x}_i$$

for the best portfolio algorithm $\mathbf{b}_i$ in each scenario is given by

$$S_n = 2^{nW^* + o(n)}.$$

References

[1] T. Cover. Universal Portfolios. *Mathematical Finance*, 1(1): 1-29, January 1991.

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