Universal Portfolios with Memory

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Abstract — **We find an adaptive finite memory portfolio selection algorithm that performs asymptotically as well as if we had known the optimal portfolio dependence on memory ahead of time.**

Summary

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be an arbitrary sequence of stock vectors in R^m, where x_{ij} denotes the price relative of the i^{th} stock on day j. Let $\mathbf{b} = (b_1, b_2, \ldots, b_m)$, $b_i \geq 0$, $\sum b_i = 1$ denote a portfolio, where b_i is the proportion of current wealth invested in stock i. If the portfolio is rebalanced to $\mathbf{b}$ each day, the resulting wealth $S_n(\mathbf{b}, \mathbf{x}^n)$ is given by

$$S_n(\mathbf{b}, \mathbf{x}^n) = \prod_{i=1}^{n} \mathbf{b}^t \mathbf{x}_i.$$

Then

$$\max_b S_n(\mathbf{b}, \mathbf{x}^n).$$

is the maximum wealth that can be achieved by a constant rebalanced portfolio given knowledge of the stock sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.

Now let the portfolio $\mathbf{b}(T)$ depend on a finite state memory

$$T_{n+1} = f(T_n, \mathbf{X}_n), \quad T_n \in \{1, 2, \ldots, d\}.$$

Here f is some fixed memory updating function. For example, the memory state T may indicate whether the market went up or down the previous day. The goal is to achieve the highest wealth achievable by portfolios $\mathbf{b}(T)$ depending only on the state T of memory at the time of the investment. This wealth S_n^* is given by

$$S_n^* = \max_{b(\cdot)} \prod_{k=1}^{n} \mathbf{b}^t(T_k)\mathbf{x}_k,$$

where the maximum is over all choices $\mathbf{b}(1), \ldots, \mathbf{b}(d)$, where $\mathbf{b}(T) = (b_1(T), \ldots, b_m(T))$, $\sum_{i=1}^m b_i(T) = 1$, $b_i(T) \geq 0$, for all memory states $T \in \{1, 2, \ldots, d\}$.

We exhibit an adaptive nonanticipating universal portfolio $\hat{\mathbf{b}}_k(T_k)$, $T_k \in \{1, 2, \ldots, d\}$, achieving wealth

$$\hat{S}_n = \prod_{k=1}^{n} \hat{\mathbf{b}}_k^t(T_k)\mathbf{x}_k$$

which tracks S_n^* in the sense that $\frac{1}{n}\log(S_n^*/\hat{S}_n) \to 0$, as $n \to \infty$, for all bounded sequences $\mathbf{x}_1, \mathbf{x}_2, \ldots$ of stock vectors. Thus the exponential growth rates of the adaptive wealth $\hat{S}_n$ and the best look-ahead wealth S_n^* agree to first order in the exponent for every sequence. The zero memory case is given in [1].

A more detailed look at the relative behavior shows that

$$\hat{S}_n \sim \left(\frac{c}{n^{(m-1)d/2}}\right) S_n^*,$$

if $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is a stationary ergodic stock market sequence, where c is a constant corresponding to the volatility of the process.

The underlying distribution of the stochastic process is effectively learned by the adaptive portfolio even when the stock sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is maliciously chosen. The penalty for universality is $\frac{1}{2}\log n$ in the exponent per degree of freedom in the portfolio selection.

This loss of $\frac{1}{2}\log n$ in the wealth exponent per degree of freedom in the portfolio parallels the loss of $\frac{1}{2}\log n$ bits per degree of freedom in universal data compression for unknown parametric sources. However, our results hold for learning any distribution; the degrees of freedom refer only to the range of possible actions (the dimensionality of the set of allowed portfolios) and not to the intrinsic dimensionality of the family of distributions as is the case for universal data compression.

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References

[1] T. Cover, Universal Portfolios, *Mathematical Finance*, 1(1):1-29, January 1991.