Capacity of A Class of Relay Channels With Orthogonal Components

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Abstract

The capacity of a class of discrete-memoryless relay channels with orthogonal channels from the sender to the relay receiver and from the sender and relay to the receiver is shown to be equal to the max-flow min-cut upper bound. The result is extended to AWGN relay channels where the channel from the sender to the relay uses a different frequency band from the channel from the sender and the relay to the receiver.

Keywords

Discrete memoryless relay channel, AWGN relay channel.

I. INTRODUCTION

The discrete-memoryless relay channel denoted by $(\mathcal{X} \times \mathcal{X}_1, p(y, y_1|x, x_1), \mathcal{Y} \times \mathcal{Y}_1)$ consists of a sender $X \in \mathcal{X}$, a receiver $Y \in \mathcal{Y}$, a relay sender $X_1 \in \mathcal{X}_1$, a relay receiver $Y_1 \in \mathcal{Y}_1$, and a family of conditional probability mass functions $p(y, y_1|x, x_1)$ on $\mathcal{Y} \times \mathcal{Y}_1$, one for each $(x, x_1) \in \mathcal{X} \times \mathcal{X}_1$. A $(2^{nR}, n)$ code for the channel consists of: (i) a set of messages $\{1, 2, \ldots, 2^{nR}\}$, (ii) an encoding function that maps each message w into a codeword $x^n(w)$ of length n, (iii) relay encoding functions $x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1(i-1)})$, for $1 \le i \le n$, and (iv) a decoding function that maps each received sequence y^n into an estimate $\hat{w}(y^n)$. A rate R is achievable if there exists a sequence of $(2^{nR}, n)$ codes with $P_e^{(n)} = \mathrm{P}\{\hat{W} \neq W\} \to 0$, as $n \to \infty$. Channel capacity C is defined as the supremum over the set of achievable rates.

The relay channel was introduced in [1]. The capacity of degraded and reversely degraded relay channels as well as upper and lower bounds on the capacity of the general relay channel were established in [2]. The capacity of the general relay channel, however, is not known.

In this note we consider the following class of discrete-memoryless relay channels with orthogonal channel components from the sender to the relay and from the sender and relay to the receiver.

Definition: A discrete-memoryless relay channel is said to have orthogonal components if the sender alphabet $\mathcal{X} = \mathcal{X}_D \times \mathcal{X}_R$ *and the channel can be expressed as* $p(y, y_1|x, x_1) = p(y|x_D, x_1)p(y_1|x_R, x_1)$, *for all* $(x_D, x_R, x_1, y, y_1) \in \mathcal{X}_D \times \mathcal{X}_R \times \mathcal{X}_1 \times \mathcal{Y} \times \mathcal{Y}_1$.

Our main result is to establish the capacity of this class of relay channels.

Theorem: The capacity of the relay channel with orthogonal components is given by

$$C = \max_{p(x_1)p(x_D|x_1)p(x_R|x_1)} \min\{I(X_D, X_1; Y), I(X_R; Y_1|X_1) + I(X_D; Y|X_1)\}. \tag{1}$$

We also extend this result to the class of AWGN relay channels where the channel from the sender to the relay uses a different frequency band from the channel from the sender and the relay to the receiver.

II. PROOF OF THEOREM

To prove the theorem we use two bounds on the capacity of the general discrete-memoryless relay channel from [2]. The first bound is the "max-flow min-cut" upper bound

$$C \le \max_{p(x,x_1)} \min\{I(X, X_1; Y), I(X; Y, Y_1|X_1)\}. \tag{2}$$

The second bound is a special case of Theorem 7 in [2], which yields the lower bound

$$C \ge \max_{p(u,x,x_1)} \min\{I(X, X_1; Y), I(U; Y_1|X_1) + I(X; Y|X_1, U)\}. \tag{3}$$

This lower bound is achieved using a *generalized block-Markov* coding scheme, where in each block the relay decodes part of the new message (represented by U) and cooperatively sends enough information with the sender (represented by X) to help the receiver decode the previous message (U then X). Note that in [3], this lower bound was shown to be optimal and equal to the max-flow min-cut upper bound for the class of semi-deterministic relay channels.

Achievability: We show that any $R < C$ is achievable using the generalized block-Markov encoding scheme. Substituting $X = (X_D, X_R)$ and $U = X_R$ in equation (3) and assuming joint probability mass function of the form $p(x_1)p(x_R|x_1)p(x_D|x_1)$, we obtain

$$\begin{aligned} I(X, X_1; Y) &= I(X_R, X_D, X_1; Y) \\ &= I(X_D, X_1; Y) + I(X_R; Y|X_D, X_1) \\ &= I(X_D, X_1; Y), \end{aligned}$$

and

$$\begin{aligned} I(U;Y_1|X_1)+I(X;Y|X_1,U) &= I(X_R;Y_1|X_1)+I(X_D,X_R;Y|X_1,X_R) \\ &\overset{(a)}{=} I(X_R;Y_1|X_1)+I(X_D;Y|X_1), \end{aligned}$$

where (a) follows by the fact that $X_R \to X_1 \to Y$ form a Markov chain.
Converse: we show that C is equal to the max-flow min-cut upper bound. Clearly

$$I(X,X_1;Y)=I(X_D,X_1;Y).$$

Next, we consider the second term under the $\min$ in equation (2)

$$\begin{aligned} I(X;Y,Y_1|X_1) &= I(X_D,X_R;Y,Y_1|X_1) \\ &= I(X_D,X_R;Y_1|X_1)+I(X_D,X_R;Y|X_1,Y_1) \\ &= I(X_R;Y_1|X_1)+I(X_D;Y_1|X_1,X_R)+I(X_D,X_R;Y|X_1,Y_1) \\ &\overset{(b)}{=} I(X_R;Y_1|X_1)+I(X_D,X_R;Y|X_1,Y_1) \\ &= I(X_R;Y_1|X_1)+H(Y|X_1,Y_1)-H(Y|X_1,X_D,X_R,Y_1) \\ &\overset{(c)}{=} I(X_R;Y_1|X_1)+H(Y|X_1,Y_1)-H(Y|X_1,X_D) \\ &\le I(X_R;Y_1|X_1)+H(Y|X_1)-H(Y|X_1,X_D) \\ &= I(X_R;Y_1|X_1)+I(X_D;Y|X_1), \end{aligned}$$

where (b) and (c) follow from the fact that $X_D \to (X_1,X_R) \to Y_1$ and $(X_R,Y_1) \to (X_1,X_D) \to Y$ each form a Markov chain, respectively. Thus we have shown that

$$C \le \max_{p(x_D,x_R,x_1)} \min\{I(X_D,X_1;Y), I(X_R;Y_1|X_1)+I(X_D;Y|X_1)\}.$$

Without loss of generality we can restrict the joint probability mass functions to be of the form $p(x_1)p(x_D|x_1)p(x_R|x_1)$. This completes the proof of the Theorem.

III. Extension to AWGN Relay Channel

The result of the Theorem can be extended to establish the capacity of the AWGN relay channel in Figure 1, where the channel from the sender to the relay uses a different frequency band from the channel from the sender and relay sender to the receiver. The additive white Gaussian noise processes $\{Z_{1i}\}$ and $\{Z_i\}$ are independent each with power N, and the constants $a, b > 0$ represent the gain of the signal over the path from the sender to the relay and from the relay to the receiver, respectively, relative to the gain of the direct channel (which is assumed to be equal to one).

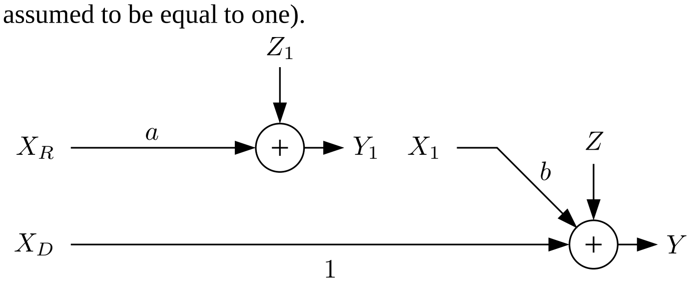

Fig. 1. Frequency-division AWGN Relay Channel.

Assuming average power constraints P on (X_D, X_R) and γP, on X_1, for $\gamma \ge 0$, the capacity of this channel is given by [1]

$$C(P,\gamma P)=\max_{0\le\alpha,\rho\le 1}\min\left\{\mathcal{C}\left(\frac{a^2(1-\alpha)P}{N}\right)+\mathcal{C}\left(\frac{\alpha(1-\rho^2)P}{N}\right),\mathcal{C}\left(\frac{(\alpha+b^2\gamma+2b\rho\sqrt{\alpha\gamma})P}{N}\right)\right\}, \qquad (4)$$

where $\mathcal{C}(x)=\frac{1}{2}\log_2(1+x)$. Again achievability is established using the generalized block-Markov scheme, where we let $X_D \sim \mathcal{N}(0,\alpha P)$ and $X_R \sim \mathcal{N}(0,(1-\alpha)P)$ be independent, and $X_1 \sim \mathcal{N}(0,\gamma P)$ be independent of X_R but jointly Gaussian with X_D with $E(X_1X_D)=\rho\sqrt{\alpha\gamma}P$. New information is sent to the relay through X_R and to the receiver through part of X_D (with power $\alpha(1-\rho^2)P$), and the relay and the rest of X_D cooperatively send information to remove the uncertainty of the receiver about the previous message.

[1] This result was reported in [4].

To prove the converse, first note that from Theorem 1, given any $(2^{nR}, n)$ sequence of codes with $P_e^{(n)} \to 0$,

$$C \le \min\{I(X_D, X_1; Y), I(X_R; Y_1|X_1) + I(X_D; Y|X_1)\}, \tag{5}$$

for some joint probability distribution $p(x_1)p(x_D|x_1)p(x_R|x_1)$. Since the channel structure in this case has the special form $p(y, y_1|x, x_1) = p(y|x_D, x_1)p(y_1|x_R)$, it follows that

$$\begin{aligned} I(X_R; Y_1|X_1) &= H(Y_1|X_1) - H(Y_1|X_1, X_R) \\ &= H(Y_1|X_1) - H(Y_1|X_R) \\ &\le H(Y_1) - H(Y_1|X_R) = I(X_R; Y_1). \end{aligned}$$

Equation (5) then reduces to

$$C \le \min\{I(X_D, X_1; Y), I(X_R; Y_1) + I(X_D; Y|X_1)\}, \tag{6}$$

for some joint distribution $p(x_1)p(x_D|x_1)p(x_R)$. Now the power cosntraints requires that $E(X_D^2) + E(X_R^2) \le P$ and $E(X_1^2) \le \gamma P$. Thus for some $0 \le \alpha \le 1$, $E(X_D^2) \le \alpha P$ and $E(X_R^2) \le (1-\alpha)P$. Define ρ to be the correlation coefficient between X_D and X_1. It is now straightforward to show that

$$I(X_D, X_1; Y) \le \mathcal{C}\left(\frac{(\alpha + b^2\gamma + 2b\rho\sqrt{\alpha\gamma})P}{N}\right),$$

The second term under the minimum in Equation (6) can be similarly upper bounded to yield

$$I(X_R; Y_1) + I(X_D; Y|X_1) \le \mathcal{C}\left(\frac{a^2(1-\alpha)P}{N}\right) + \mathcal{C}\left(\frac{\alpha(1-\rho^2)P}{N}\right).$$

This completes the proof of converse.

IV. Conclusion

The paper establishes the capacity of the class of relay channels with orthogonal channels from the sender to the relay and from the sender and relay to the receiver. As for all other classes of relay channels with known capacity [2, 3], the capacity for this class is equal to the max-flow min-cut upper bound.

Our result points to the main difficulty in establishing the capacity of the general relay channel, which is finding the optimal broadcasting strategy from the sender to the relay Y_1 and the receiver Y. For the class of relay channels discussed here, the optimal strategy is to split the message into two parts, one is decoded by the relay and sent cooperatively with the sender to the receiver and the other is sent directly to the receiver. This strategy, however, is not optimal in general. In [5], it was shown that for AWGN relay channels, even when the channel from the sender to both the relay and receiver is orthogonal to the channel from the relay to the receiver, other strategies such as linear relaying and side information coding can achieve higher rates.

References

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