Randomized Algorithms for Solving Linear Systems with Low-Rank Structure

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Based on joint works with Zachary Frangella, Daniel LeJeune, Christopher Musco, Deanna Needell, Pratik Rathore, Elizaveta Rebrova, Aaron Sidford, Madeleine Udell, and Jiaming Yang

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Outline

Example. Solve this system of **linear equations**:

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned}$$

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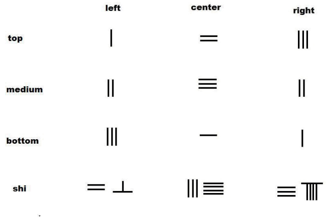
Solution: **Method of elimination** (a.k.a. Gaussian elimination)

First appeared in:
The Nine Chapters on the Mathematical Art,
China, 2nd century

Later formalized by:
Newton, and then Gauss, among others.



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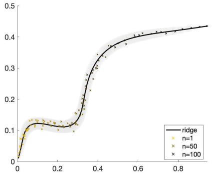
Solving linear systems in modern applications

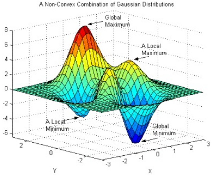
Statistical inference

$$y_i = f^*(\mathbf{x}_i) + \xi_i, \qquad f^* \in \mathcal{F}, \quad (\mathbf{x}_i, y_i) \sim \mathcal{D}.$$



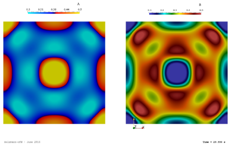
Nonlinear optimization

$$\text{Minimize} \quad f(\mathbf{x}) = \sum_{i=1}^{n} \log(1 + \mathrm{e}^{-y_i \mathbf{a}_i^\top \mathbf{x}_i})$$



Partial differential equations

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 + F(1-u)$$



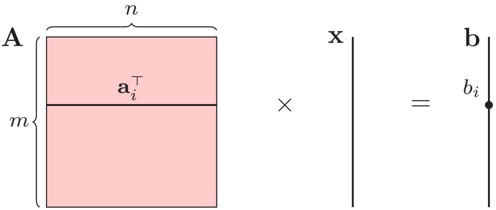
What is the cost of solving an $m \times n$ linear system?

Answer: It depends...

- ... on the sparsity of the input matrix,
- ... on its singular value decay profile,
- ... on whether it has domain-specific structural properties (e.g., Laplacian, Hankel, Toeplitz, circulant matrices, etc.),
- ... on whether we need exact, or high/machine precision, or medium/low precision,
- ... on whether we can tolerate a small chance of failure.

Two perspectives on solving linear systems

Task: Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^n$, solve a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.
For simplicity, assume that the system is consistent and $m = O(n)$.



Direct methods

- Method of elimination
- QR, LU, SVD, ...

$O(n^3)$ time

for "ill-conditioned systems"

Iterative methods

- Conjugate gradient (CG)
- MINRES, GMRES, LSQR, ...

$O(n^2)$ time × T iterations

for "well-conditioned systems"

Can we unify the two perspectives?

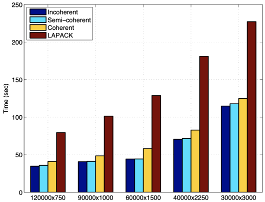
Key ingredient: Randomization

Randomization

Using **randomized algorithms** to solve **deterministic problems**.

Early successes in NLA [AMT10]

*“**beats LAPACK’s** direct dense **least-squares** solver …on essentially any dense **tall matrix.**”*



Moving towards “RandLAPACK”

Murray et al. *“Randomized Numerical Linear Algebra: A Perspective on the Field with an Eye to Software,”* arXiv:2302.11474, 2023.

[AMT10] Avron, Maymounkov, and Toledo. “Blendenpik: Supercharging LAPACK’s least-squares solver”, *SIAM Journal on Scientific Computing* 32.3: 1217-1236, 2010.

This talk: Randomization for general linear systems

- Low-rank structure in ill-conditioned systems

  In "typical" systems, a low-rank component causes ill-conditioning.

- Randomized algorithms for low-rank structure

  Randomized sketching-based methods can go beyond traditional (Krylov-based) methods for solving low-rank structured systems.

- Case study: Solving a dense $10^8 \times 10^8$ linear system

  Scaling full Kernel Ridge Regression to massive datasets.

- Unified perspective on the complexity of solving linear systems:

$$\tilde{O}\Big(\underbrace{k^3}_{\text{low-rank}} + \underbrace{n^2}_{\text{well-conditioned}} \Big)$$

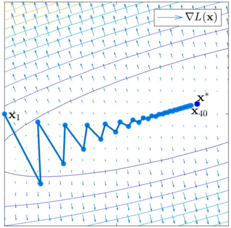
Outline

Convergence of iterative methods

Task: Given $\mathbf{A} \in \mathbb{R}^{O(n)\times n}$ and $\mathbf{b} \in \mathbb{R}^n$, solve a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$

Iterative methods

- Conjugate gradient (CG)
- MINRES, GMRES, LSQR, ...



$$\underbrace{\text{Cost of } \mathbf{v} \to \mathbf{A}\mathbf{v}}_{O(n^2) \text{ operations}} \quad \times \quad T \text{ iterations}$$

How does the number of iterations T depend on $\mathbf{A}$?

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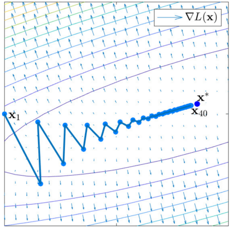


$\underbrace{\text{Cost of } \mathbf{v} \to \mathbf{Av}}_{O(n^2) \text{ operations}} \quad \times \quad T \text{ iterations}$

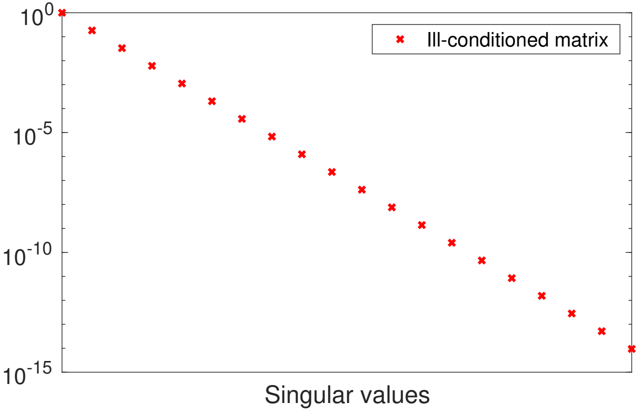
How does the number of iterations T depend on $\mathbf{A}$?

Naïve answer: T scales with the *condition number*, $\kappa = \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})}$.

$\Rightarrow$ *"Iterative methods perform poorly for ill-conditioned systems."*

Typical ill-conditioned system

What does the spectrum of a "typical" ill-conditioned matrix look like?



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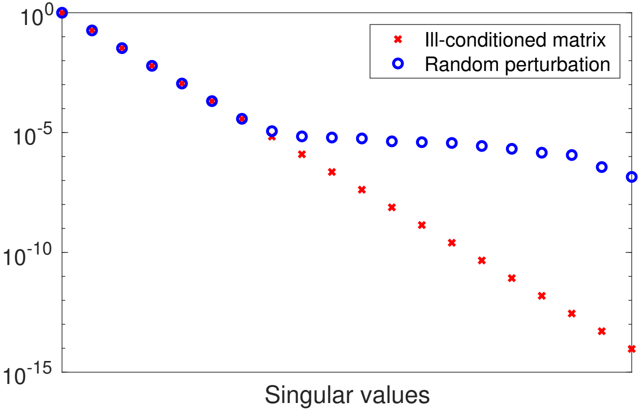
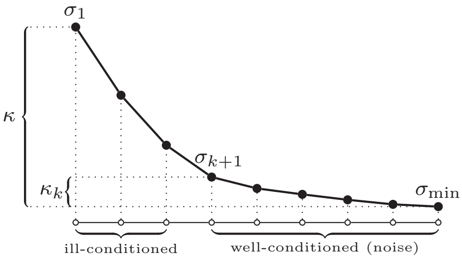


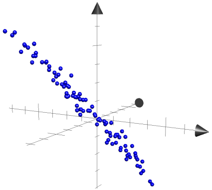
Random perturbation: $\tilde{\mathbf{A}} = \mathbf{A} + \mathbf{G}, \qquad \|\mathbf{G}\| \leq 10^{-5}\|\mathbf{A}\|$

Model: Systems with low-rank structure

Low-rank structure: Implicitly partition the spectrum of $\mathbf{A}$

1. Ill-conditioned top-k: favors *direct* methods
2. Well-conditioned tail: favors *iterative* methods





Systems with low-rank structure are ubiquitous across many areas!

- "Signal + noise" data, e.g., smoothed analysis, stochastic rounding, ...
- Deliberate regularization in ML/Stats/Opt, e.g., $\mathbf{A} = \mathbf{B} + \lambda\mathbf{I}$
- Key subroutine in matrix norm and eigenvalue estimation methods

Back to convergence of iterative methods

Task: Given $\mathbf{A} \in \mathbb{R}^{O(n)\times n}$ and $\mathbf{b} \in \mathbb{R}^n$, solve a linear system $\mathbf{Ax} = \mathbf{b}$

Iterative methods

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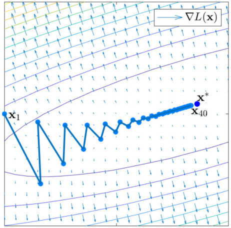

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How do they perform on ***systems with low-rank structure****?*

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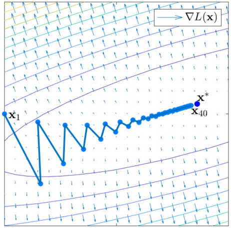

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Answer: Convergence theory of **Krylov Subspace Methods**

Krylov subspaces and polynomial approximation

Definition: Given square matrix $\mathbf{A}$ and vector $\mathbf{b}$, the order-k Krylov subspace is defined as:

$$\mathcal{K}_k(\mathbf{A}, \mathbf{b}) = \text{span}\left\{\mathbf{b},\ \mathbf{A}\mathbf{b},\ \mathbf{A}^2\mathbf{b},\ \ldots,\ \mathbf{A}^{k-1}\mathbf{b}\right\}.$$

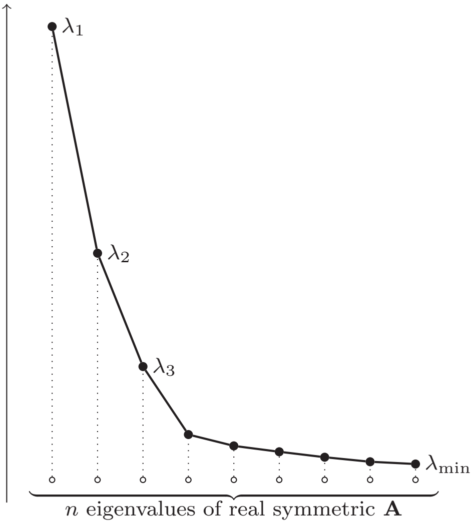
Property: Any vector $\mathbf{v} \in \mathcal{K}_k(\mathbf{A}, \mathbf{b})$ can be expressed as $\mathbf{v} = p(\mathbf{A})\mathbf{b}$, where $p(x) = c_0 + c_1 x \ldots + c_{k-1} x^{k-1}$ is a polynomial of degree $k-1$.

Recipe for linear solver: Gradually build a Krylov subspace and maintain "best" approximation $\hat{\mathbf{x}} = p(\mathbf{A})\mathbf{b}$ for $\mathbf{A}^{-1}\mathbf{b}$ in that subspace.

Dominant cost: Matrix-vector product to compute the next direction,

$$\mathbf{A}^k\mathbf{b} = \mathbf{A} \cdot (\mathbf{A}^{k-1}\mathbf{b}) \quad \text{in } O(n^2) \text{ arithmetic operations.}$$

Krylov subspaces and eigenvalue clusters



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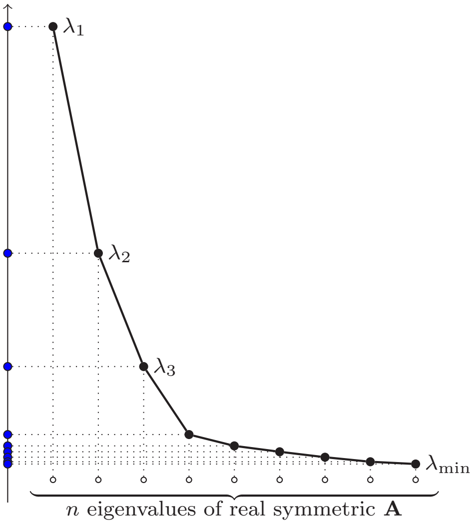

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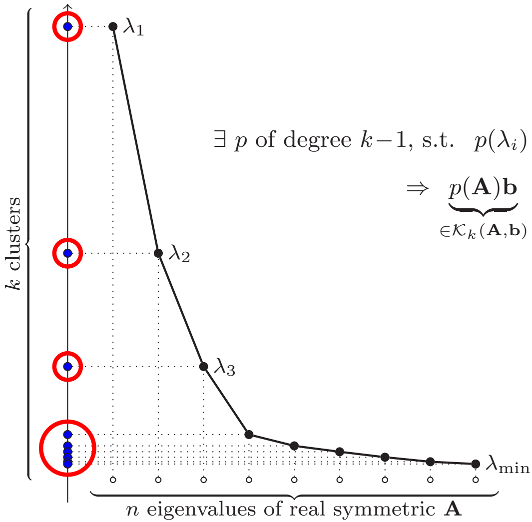
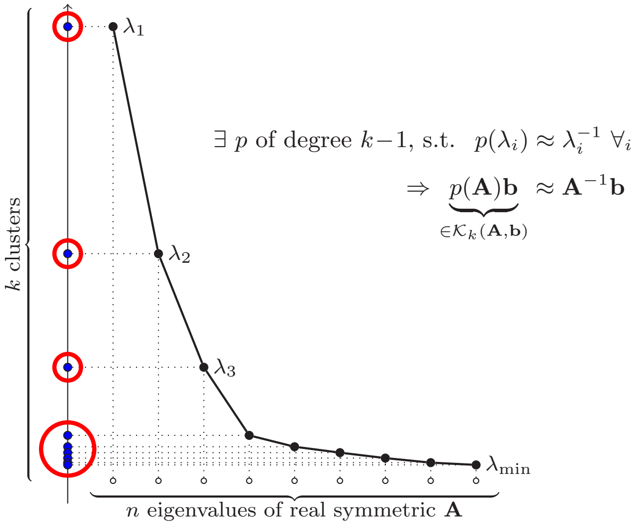


$$\exists\ p \text{ of degree } k-1, \text{ s.t. } \quad p(\lambda_i) \approx \lambda_i^{-1}\ \forall_i$$

$$\Rightarrow \underbrace{p(\mathbf{A})\mathbf{b}}_{\in \mathcal{K}_k(\mathbf{A},\mathbf{b})} \approx \mathbf{A}^{-1}\mathbf{b}$$

Krylov subspaces and eigenvalue clusters



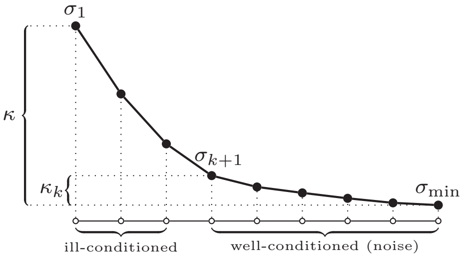
Conclusion: If the eigenvalues of $\mathbf{A}$ form k clusters, then $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ contains an accurate approximation to $\mathbf{x}^* = \mathbf{A}^{-1}\mathbf{b}$.

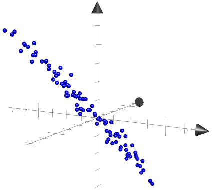
Krylov subspace methods: Convergence

Theorem ([AL86])

If $\mathbf{A}$ *has singular values* $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min}$ *with* $\kappa_k = \frac{\sigma_{k+1}}{\sigma_{\min}}$, *there is a Krylov subspace method (e.g., LSQR) that finds an* ϵ*-approximate solution* $\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\| \leq \epsilon\|\mathbf{b}\|$ *in*

$$T = O(k + \kappa_k \log 1/\epsilon) \quad \textit{iterations.}$$





(We use singular values instead of eigenvalues to capture the non-symmetric case.)

[AL86] Axelsson and Lindskog. "On the rate of convergence of the preconditioned conjugate gradient method," *Numerische Mathematik*. 48:499-523, 1986.

Krylov subspace methods and low-rank structure

Conclusion: To solve an $O(n) \times n$ linear system with at most k large isolated singular values and a cluster of width κ_k, we need:

$$\underbrace{\text{Cost of } \mathbf{v} \to \mathbf{A}\mathbf{v}}_{O(n^2) \text{ operations}} \times O(k + \kappa_k \log 1/\epsilon) \;=\; O(n^2 k + n^2 \kappa_k \log 1/\epsilon)$$

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Question: Can we avoid the $n^2 k$ bottleneck in Krylov methods?

Krylov subspace methods and low-rank structure

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Yes! and **No!**

With only $\mathbf{v} \to \mathbf{A}\mathbf{v}$ access to $\mathbf{A}$,

Krylov methods are optimal.

Krylov subspace methods and low-rank structure

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Question: Can we avoid the $n^2 k$ bottleneck in Krylov methods?

Yes! and **No!**

With *direct* access to $\mathbf{A}$, randomized methods do better.

With only $\mathbf{v} \to \mathbf{A}\mathbf{v}$ access to $\mathbf{A}$, Krylov methods are optimal.

Our result: $\tilde{O}(k^3 + n^2 \kappa_k \log 1/\epsilon)$

Outline

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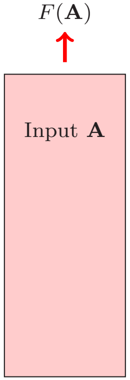
Main tool: Randomized sketching

Sketching operator: Random dimension reducing linear map (matrix $\mathbf{S}$)
E.g.: Gaussian/sparse matrices, randomized Hadamard transforms, ...



Crucially: Can be *much* faster than dense matrix multiplication

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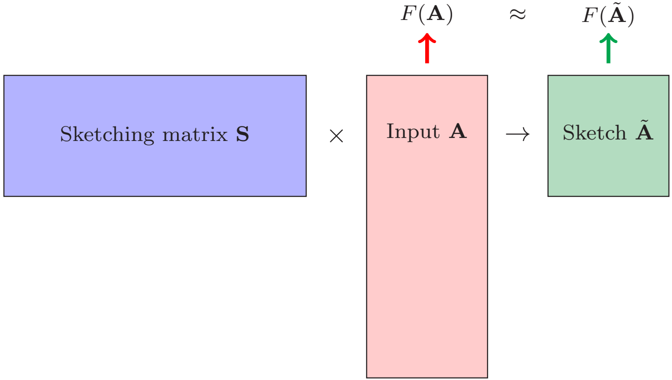

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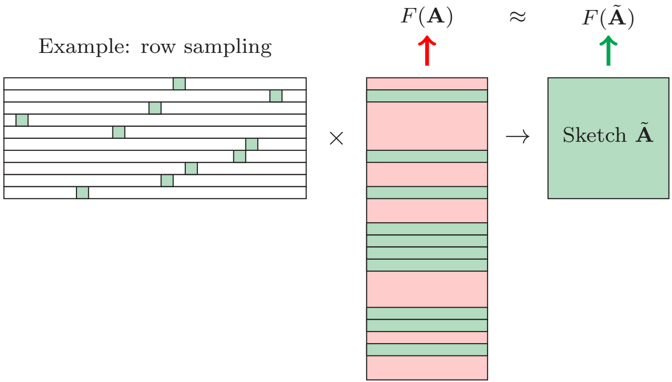

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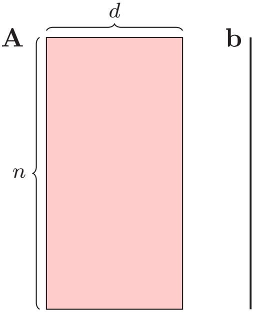
Example: Least squares

Over-determined linear system: Many more equations than unknowns, also known as *least squares*.



$$\text{Compute} \quad \mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$$

$$\text{for} \quad \mathbf{A} \in \mathbb{R}^{n \times d}, \ \mathbf{b} \in \mathbb{R}^n$$

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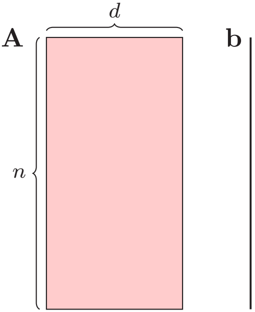
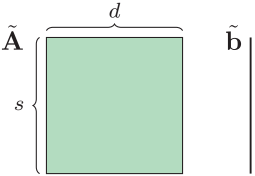


Sketching leverages this redundancy:

Compute $\widetilde{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\tilde{\mathbf{A}}\mathbf{x} - \tilde{\mathbf{b}}\|_2^2$

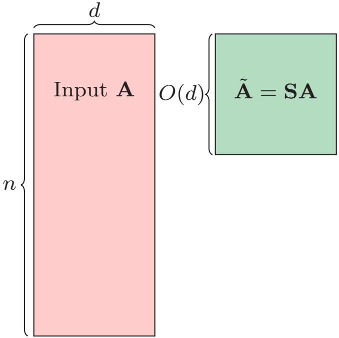
for $\underbrace{\tilde{\mathbf{A}} = \mathbf{S}\mathbf{A}}_{\text{sketch of } \mathbf{A}}, \ \underbrace{\tilde{\mathbf{b}} = \mathbf{S}\mathbf{b}}_{\text{sketch of } \mathbf{b}}$



Why does sketching work for least squares?

Fact. In $\tilde{O}(nd)$ time, we can compute $\tilde{\mathbf{A}} \in \mathbb{R}^{O(d) \times d}$ such that for $i = 1, ..., d$:

$$\sigma_i^2(\tilde{\mathbf{A}}) = (1 \pm \tfrac{1}{2})\sigma_i^2(\mathbf{A})$$



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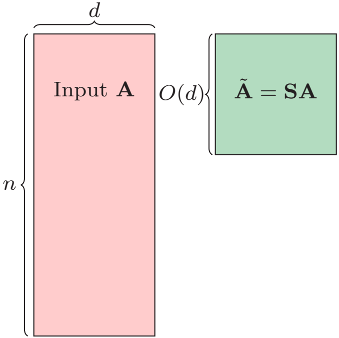
Even better. This preserves the whole covariance structure of the matrix:

$$\tilde{\mathbf{A}}^\top \tilde{\mathbf{A}} = (1 \pm \tfrac{1}{2})\mathbf{A}^\top \mathbf{A}.$$



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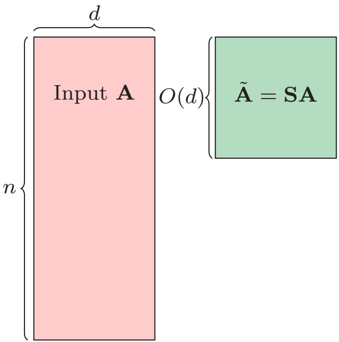
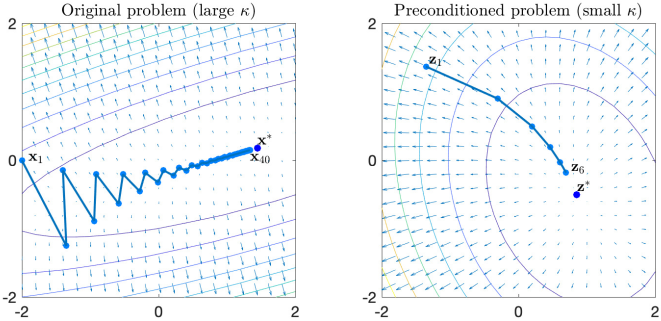


Fast least squares solver: Runs in $\tilde{O}(d^3 + nd)$ time

1. Rewrite least squares via normal equations, $\mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{b}$
2. Precondition your favorite iterative method with $\tilde{\mathbf{A}}^\top \tilde{\mathbf{A}}$

Fast least squares solver

<u>Compute preconditioner:</u> $(\tilde{\mathbf{A}}^\top \tilde{\mathbf{A}})^{-1} \approx (\mathbf{A}^\top \mathbf{A})^{-1}$ Cost: $O(d^3)$



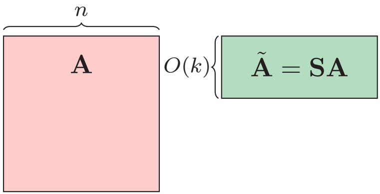
$$\mathbf{A}^\top \mathbf{A} \mathbf{x} = \mathbf{A}^\top \mathbf{b}$$

$\kappa(\mathbf{A}^\top \mathbf{A})$ is large

$$\mathbf{A}^\top \mathbf{A} (\tilde{\mathbf{A}}^\top \tilde{\mathbf{A}})^{-1} \mathbf{z} = \mathbf{b}$$

$\kappa(\mathbf{A}^\top \mathbf{A} (\tilde{\mathbf{A}}^\top \tilde{\mathbf{A}})^{-1})$ is small

Does this work for general linear systems?



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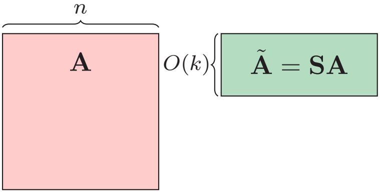

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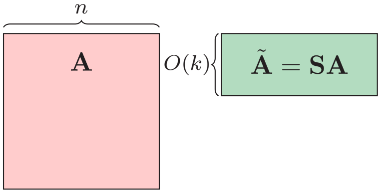

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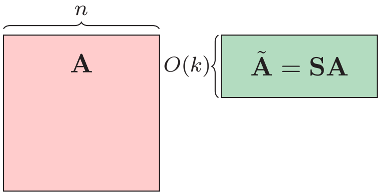

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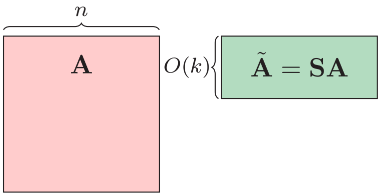

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Can we still produce a fast solver via sketching?

Yes! Just use multiple sketches:

1. Sketch-and-Project
2. Recursive Sketching

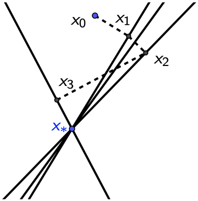
Outline

Background: The Kaczmarz algorithm

Idea: Iteratively project onto the solutions of individual equations.

Starting at $\mathbf{x}_0$, for $t = 0, 1, 2, \ldots$

1. Select index I_t
2. Project current iterate $\mathbf{x}_t$ onto the solutions of I_t-th equation



Randomized Kaczmarz: Select indices via weighted sampling [SV09]

The first Kaczmarz algorithm with provable convergence rate.

[Kac37] Stefan Kaczmarz, "Angenäherte Auflösung von Systemen linearer Gleichungen", *Bulletin International de l'Académie Polonaise des Sciences et des Lettres* 35:355–357, 1937.

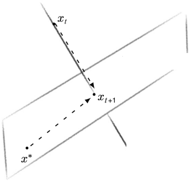
[SV09] Strohmer and Vershynin, "A randomized Kaczmarz algorithm with exponential convergence", *Journal of Fourier Analysis and Applications*, 14.2:262-278, 2009.

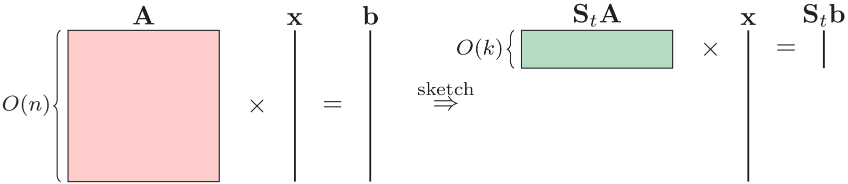
Powerful extension: Sketch-and-Project

Starting at $\mathbf{x}_0 \in \mathbb{R}^n$, for $t = 0, 1, 2, ...$

1. Sample random $O(k) \times O(n)$ matrix $\mathbf{S}_t$.
2. Project $\mathbf{x}_t$ onto the solutions of $\mathbf{S}_t\mathbf{A}\mathbf{x} = \mathbf{S}_t\mathbf{b}$:

$$\mathbf{x}_{t+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{x}_t - \mathbf{x}\|^2 \quad \text{subject to} \quad \mathbf{S}_t\mathbf{A}\mathbf{x} = \mathbf{S}_t\mathbf{b}.$$





[GR15] Gower and Richtárik, "Randomized iterative methods for linear systems", *SIAM Journal on Matrix Analysis and Applications*, 36.4:1660-1690, 2015.

Key insight: Sharp analysis of Sketch-and-Project

Theorem ([DR24])

If $\mathbf{S}_t$ is a Gaussian matrix, then Sketch-and-Project satisfies:

$$\mathbb{E}\,\|\mathbf{x}_t - \mathbf{x}^*\|^2 \;\leq\; \left(1 - \frac{\sigma_{\min}^2(\mathbf{A})}{\frac{1}{k}\sum_{i>k}\sigma_i^2(\mathbf{A})}\right)^t \|\mathbf{x}_0 - \mathbf{x}^*\|^2.$$

[DR24] Dereziński and Rebrova, "Sharp Analysis of Sketch-and-Project Methods via a Connection to Randomized Singular Value Decomposition", *SIAM Journal on Mathematics of Data Science*, 6.1:127-153, 2024.

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- Each sketch-and-project step runs on a $\frac{k}{n}$**-fraction** of the data
- This runtime gain should cancel out the **tail noise** $\leq \frac{n}{k}\sigma_{k+1}^2$

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Hang on, Gaussian sketching is still too expensive!

[DR24] Dereziński and Rebrova, "Sharp Analysis of Sketch-and-Project Methods via a Connection to Randomized Singular Value Decomposition", *SIAM Journal on Mathematics of Data Science*, 6.1:127-153, 2024.

Making this efficient

Advances in sketch-and-project for systems with low-rank structure:

1. Sketching: *Gaussian guarantees for fast sketches*
   - Randomized Hadamard transform [DY24]
   - Leverage score sampling [RFY$^+$25]

2. Projecting: *Fast computation of the projection step*
   - Fast inner solver using PCG [DY24]
   - Amortizing projection cost across iterations [DNRY25]

3. Acceleration: *Improved convergence using momentum*
   - Convergence analysis with Nesterov's momentum [DLNR24]
   - Adaptive tuning of momentum parameters [DNRY25]

Making this efficient: Sketch-and-Project++

Theorem ([DNRY25])

Given any k, we can solve an $O(n) \times n$ linear system $\mathbf{Ax} = \mathbf{b}$ in time:

$$\tilde{O}\big(nk^2 + n^2\kappa_k \log 1/\epsilon\big), \quad \textit{where} \quad \kappa_k = \frac{\sigma_{k+1}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})}.$$

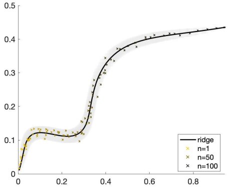
- Overcomes the n^2k bottleneck of Krylov subspace methods.
- Sketch-and-Project++: New family of randomized linear solvers
  Implementations: Kaczmarz++/CD++ [DNRY25], ASkotch [RFY$^+$25]

(Still does not attain the promised $k^3 + n^2$ guarantee...)

[DNRY25] Dereziński, Needell, Rebrova, and Yang. *"Randomized Kaczmarz methods with beyond-Krylov convergence."* arXiv:2501.11673, 2025.

[RFY$^+$25] Rathore, Frangella, Yang, Dereziński, and Udell. *"Have ASkotch: Fast cocktails for large-scale kernel ridge regression."* arXiv:2407.10070, 2025.

Case study: Large-scale Kernel Ridge Regression



Task: Fitting non-linear functions $f : \mathcal{X} \to \mathbb{R}$:

$$\min_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \Big(f(\phi_i) - y_i \Big)^2 + \frac{\lambda}{2} \|f\|_{\mathcal{H}}^2$$

When $\mathcal{H}$ is a reproducing kernel Hilbert space defined by $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, this reduces to solving an $n \times n$ linear system:

$$\underbrace{(\mathbf{K} + n\lambda \mathbf{I})}_{\text{low-rank structure}} \mathbf{x} \;=\; \mathbf{y}, \qquad \text{where} \quad \mathbf{K} = \big[k(\phi_i, \phi_j)\big]_{i,j}.$$

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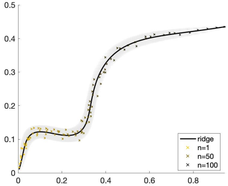

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What if we are given 100 *million data points* (ϕ_i, y_i)*?*

Solving a dense $10^8 \times 10^8$ linear system

Solving KRR for New York City taxi transportation data ($n = 10^8$)

1. Storage: $40,000$TB (terabytes) to store $\mathbf{K}$ in single precision
2. Compute: State-of-the-art solvers take $>$ 24h for single iteration

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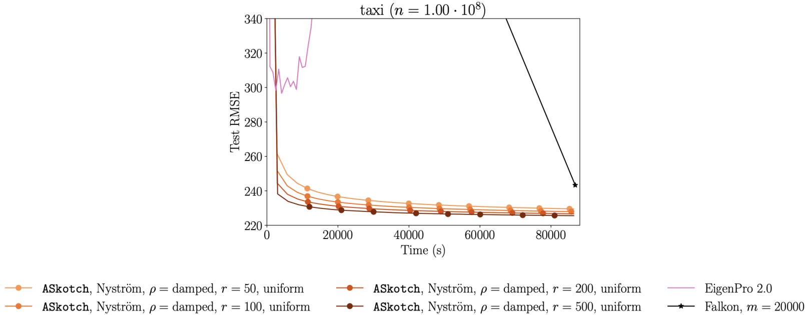
We attack the original problem with a provably convergent solver!

Solving a dense $10^8 \times 10^8$ linear system

Method: ASkotch ("Sketch-and-Project++" developed for GPUs)

Baselines: EigenPro 2.0 [MB19] and Falkon [RCR17]

Test RMSE: Root mean squared error on the test set.



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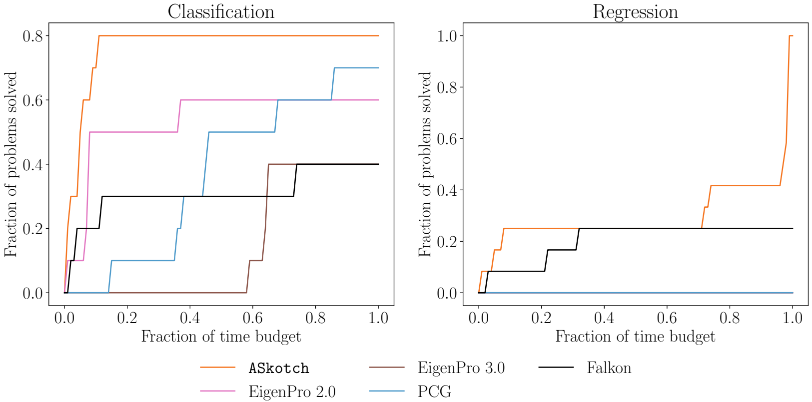
Experiment: 23 tasks, including particle physics (4 datasets) and computational chemistry (9 datasets), with dimensions at least 10^5.



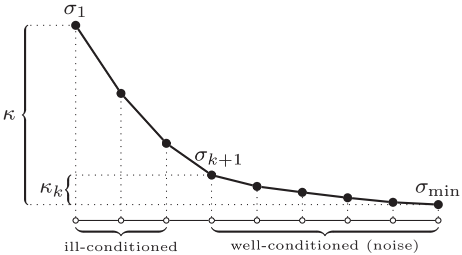
The code is available at `https://github.com/pratikrathore8/fast_krr`.

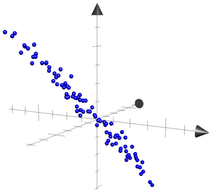
Outline

Roadmap

Complexity of solving linear systems with low-rank structure:

$$\text{Krylov methods:} \qquad \tilde{O}\left(n^2 k \;+\; n^2 \kappa_k \right)$$





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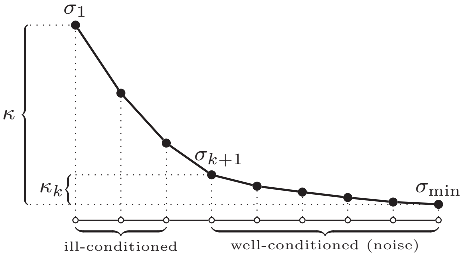

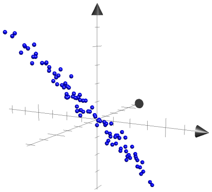

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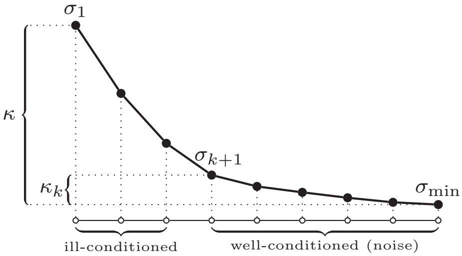
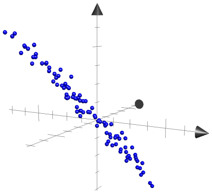
Recursive Sketching: $\tilde{O}\left(k^3 + n^2 \kappa_k \right)$





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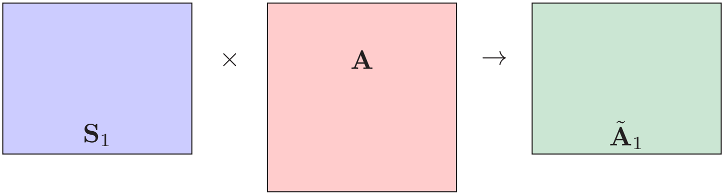

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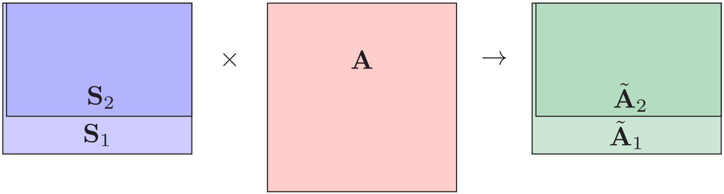

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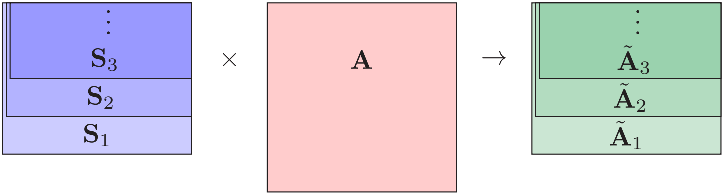

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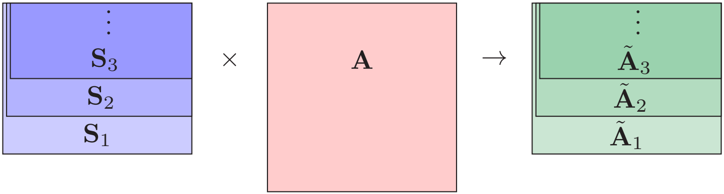

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Inspired by domain-specific preconditioning techniques:

- Recursive solvers for graph Laplacians in theoretical computer science
- Multigrid solvers for differential equations in scientific computing

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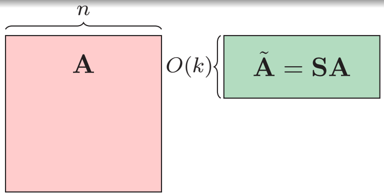

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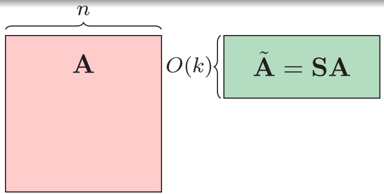

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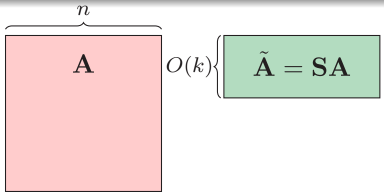

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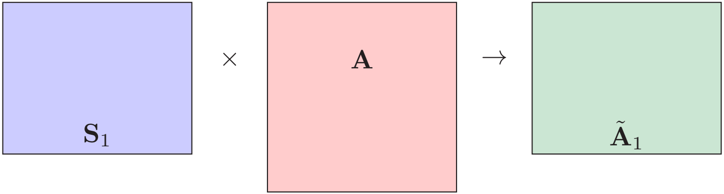
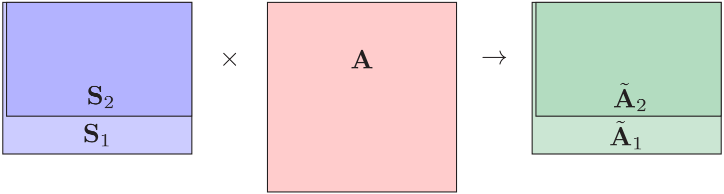
Wait! Are we allowed to do that?

Strategy: Gradually impose stricter low-rank structure



$$
\begin{aligned}
\mathbf{A}^\top\mathbf{A} \;\approx\; & \mathbf{A}^\top\mathbf{A} + \sigma_n^2\mathbf{I} \\
& \wr\wr \\
& \tilde{\mathbf{A}}_1^\top\tilde{\mathbf{A}}_1 + \lambda_1\mathbf{I}, \qquad \lambda_1 > \sigma_n^2
\end{aligned}
$$

Strategy: Gradually impose stricter low-rank structure



$$\mathbf{A}^\top \mathbf{A} \;\approx\; \mathbf{A}^\top \mathbf{A} + \sigma_n^2 \mathbf{I}$$

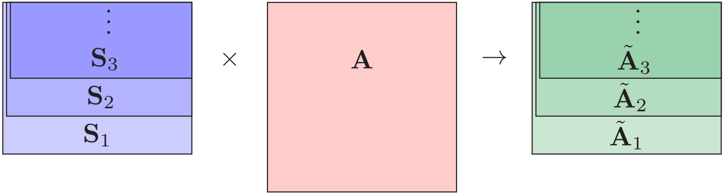
$$\wr\wr$$

$$\tilde{\mathbf{A}}_1^\top \tilde{\mathbf{A}}_1 + \lambda_1 \mathbf{I}, \qquad \lambda_1 > \sigma_n^2$$

$$\wr\wr$$

$$\tilde{\mathbf{A}}_2^\top \tilde{\mathbf{A}}_2 + \lambda_2 \mathbf{I}, \qquad \lambda_2 > \lambda_1$$

Strategy: Gradually impose stricter low-rank structure



$$
\begin{array}{llll}
\mathbf{A}^\top\mathbf{A} & \approx & \mathbf{A}^\top\mathbf{A} + \sigma_n^2\mathbf{I} & \\
 & & \wr\wr & \\
 & & \tilde{\mathbf{A}}_1^\top\tilde{\mathbf{A}}_1 + \lambda_1\mathbf{I}, & \lambda_1 > \sigma_n^2 \\
 & & \wr\wr & \\
 & & \tilde{\mathbf{A}}_2^\top\tilde{\mathbf{A}}_2 + \lambda_2\mathbf{I}, & \lambda_2 > \lambda_1 \\
 & & \wr\wr & \\
 & & \tilde{\mathbf{A}}_3^\top\tilde{\mathbf{A}}_3 + \lambda_3\mathbf{I}, & \lambda_3 > \lambda_2 \\
 & & \vdots &
\end{array}
$$

Recursive Sketching: Main result

Theorem ([DS25])

Given any k, we can solve an $O(n) \times n$ linear system $\mathbf{Ax} = \mathbf{b}$ in time:

$$\tilde{O}\big(k^3 + n^2 \kappa_k \log 1/\epsilon\big), \quad \textit{where} \quad \kappa_k = \tfrac{\sigma_{k+1}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})}.$$

- Natural complexity limit for systems with low-rank structure
- Even better: κ_k can be replaced by a *smoothed* condition number,

$$\bar{\kappa}_k = \frac{1}{n-k} \sum_{i>k} \frac{\sigma_i}{\sigma_{\min}} < \kappa_k$$

Application: The first algorithm for approximating the nuclear norm $\|\mathbf{A}\|_1 = \sum_i \sigma_i$ of an $n \times n$ matrix in nearly linear time $\tilde{O}(n^2)$.

[DS25] Dereziński and Sidford. *"Approaching Optimality for Solving Dense Linear Systems with Low-Rank Structure."* arXiv:2507.11724, 2025.

Outline

Conclusions

Linear systems with low-rank structure: a natural model for ill-conditioned matrices, e.g., in ML/Opt/Stats applications.

Randomized algorithms:

- *Sketch-and-Project++*: New family of randomized linear solvers that scaled to massive problem sizes.
- *Recursive Sketching*: Attains nearly optimal complexity for solving dense linear systems with low-rank structure.

Next steps:

- *Extending to sparse matrices and other access models*
- *Extending to other natural singular/eigenvalue profiles*

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Pseudocode for Kaczmarz++

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Input: $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, B, k, $\mathbf{x}_0$, $t_{\max}$, η, ρ, λ

$\mathbf{D} \rightarrow \mathrm{diag}(\text{random } \pm 1/\sqrt{m})$

$\mathbf{A} \leftarrow \mathbf{HDA}$ and $\mathbf{b} \leftarrow \mathbf{HDb}$ ▷ Randomized Hadamard

$\mathbf{m}_0 \leftarrow \mathbf{0}$;

Sample $\mathcal{B} = \{S_1, ..., S_B\} \subseteq \binom{[m]}{k}$ ▷ Random blocks

for $t = 0, 1, \ldots, (t_{\max} - 1)$ **do**

  Draw a random subset S from $\mathcal{B}$ ▷ Fast sketching

  $\mathbf{r}_t \leftarrow \mathbf{A}_S \mathbf{x}_t - \mathbf{b}_S$

  $\mathbf{w}_t \approx \mathrm{argmin}_{\mathbf{w}} \left\{ \|\mathbf{A}_S \mathbf{w} - \mathbf{r}_t\|^2 + \lambda \|\mathbf{w}\|^2 \right\}$ ▷ Fast projection

  $\mathbf{m}_{t+1} \leftarrow \frac{1-\rho}{1+\rho}(\mathbf{m}_t - \mathbf{w}_t)$

  $\mathbf{x}_{t+1} \leftarrow \mathbf{x}_t - \mathbf{w}_t + \eta \mathbf{m}_t$ ▷ Momentum acceleration

end for

return $\tilde{\mathbf{x}} = \mathbf{x}_{t_{\max}}$;

The code is available at `https://github.com/EdwinYang7/kaczmarz-plusplus`