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The Capacity Region of the Discrete Memoryless Interference Channel with Strong Interference

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Abstract —The capacity region of the discrete memoryless interference channel with strong interference is established.

I. INTRODUCTION

The discrete memoryless interference channel with strong interference is a discrete memoryless interference channel with inputs X_1 and X_2 and corresponding outputs Y_1 and Y_2 which satisfy

$$I(X_1;Y_1|X_2) \le I(X_1;Y_2|X_2) \tag{1}$$

and

$$I(X_2;Y_2|X_1) \le I(X_2;Y_1|X_1) \tag{2}$$

for all product probability distributions on $\mathcal{X}_1 \times \mathcal{X}_2$.

In [1] Sato conjectures that the capacity region of this channel coincides with the capacity region C of the model where both messages are required at both receiving terminals [2]. This region can be expressed as the union of the rate pairs (R_1, R_2) satisfying

$$0 \le R_1 \le I(X_1;Y_1|X_2,Q) \tag{3}$$

$$0 \le R_2 \le I(X_2;Y_2|X_1,Q) \tag{4}$$

$$R_1 + R_2 \le \min\{I(X_1,X_2;Y_1|Q), I(X_1,X_2;Y_2|Q)\} \tag{5}$$

where Q is a time-sharing parameter of cardinality 4, and the union is over all probability distributions of the form $p(q)p(x_1|q)p(x_2|q)p(y_1,y_2|x_1,x_2)$, with $p(y_1,y_2|x_1,x_2)$ set by the channel.

We prove Sato's conjecture using a result by Körner and Marton [3] that involves the notion of more capable broadcast channels. A conveniently modified version of this result is stated in Section III and proved in the Appendix.

II. PRELIMINARIES

Let W_1 and W_2 be two independent information sources uniformly distributed over the integer sets $\{1,\cdots,M_1\}$ and $\{1,\cdots,M_2\}$, respectively. Encoder 1 maps W_1 into codeword $\mathbf{X}_1$ and encoder 2 maps W_2 into codeword $\mathbf{X}_2$. The interference channel consists of four finite alphabets $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$, and $\mathcal{Y}_2$, and conditional probability distributions $p(y_1|x_1,x_2)$ and $p(y_2|x_1,x_2)$. An (M_1, M_2, n, λ_n)-code for this channel is a set of two encoding functions e_1: $M_1 \to \mathcal{X}_1^n$, e_2: $M_2 \to \mathcal{X}_2^n$ and two decoding functions d_1: $\mathcal{Y}_1^n \to M_1$, d_2: $\mathcal{Y}_2^n \to M_2$ such that

$$\lambda_{1,n} = \frac{1}{M_1M_2}\sum_{w_1,w_2} P(d_1(\mathbf{Y}_1) \ne w_1|W_1 = w_1, W_2 = w_2) \tag{6}$$

$$\lambda_{2,n} = \frac{1}{M_1M_2}\sum_{w_1,w_2} P(d_2(\mathbf{Y}_2) \ne w_2|W_1 = w_1, W_2 = w_2) \tag{7}$$

$$\max\{\lambda_{1,n},\lambda_{2,n}\} \triangleq \lambda_n. \tag{8}$$

A rate pair (R_1, R_2) is said to be achievable if there is a sequence of $(2^{nR_1}, 2^{nR_2}, n, \lambda_n)$-codes with $\lambda_n \to 0$ as $n \to \infty$. The capacity region of the interference channel is defined as the closure of the set of all achievable rate pairs.

III. ACHIEVABILITY AND CONVERSE

The achievability of the rate pairs in C is immediate since C is the capacity region when both messages W_1 and W_2 are required at both receivers [2]. Inequalities (3) and (4) represent obvious upper bounds on the rates R_1 and R_2. Therefore, by symmetry, to establish Sato's conjecture it suffices to show

$$R_1 + R_2 \le I(X_1,X_2;Y_2|Q). \tag{9}$$

From Fano's inequality, we have

$$H(W_1|\mathbf{Y}_1) \le nR_1\lambda_{1,n} + h(\lambda_{1,n}) \triangleq n\epsilon_{1,n} \tag{10}$$

$$H(W_2|\mathbf{Y}_2) \le nR_2\lambda_{2,n} + h(\lambda_{2,n}) \triangleq n\epsilon_{2,n} \tag{11}$$

where $h(\cdot)$ is the binary entropy function and $\epsilon_{1,n}$, $\epsilon_{2,n} \to 0$ as $\lambda_n \to 0$. Now consider

$$n(R_1+R_2) = H(W_1) + H(W_2)$$
$$= I(W_1;\mathbf{Y}_1) + I(W_2;\mathbf{Y}_2) + H(W_1|\mathbf{Y}_1) + H(W_2|\mathbf{Y}_2).$$

Using Fano's inequality with $\epsilon_n = \max\{\epsilon_{1,n}, \epsilon_{2,n}\}$, we get

$$n(R_1+R_2) \le I(W_1;\mathbf{Y}_1) + I(W_2;\mathbf{Y}_2) + 2n\epsilon_n \tag{12}$$

$$\le I(\mathbf{X}_1;\mathbf{Y}_1) + I(\mathbf{X}_2;\mathbf{Y}_2) + 2n\epsilon_n \tag{13}$$

$$\le I(\mathbf{X}_1;\mathbf{Y}_1|\mathbf{X}_2) + I(\mathbf{X}_2;\mathbf{Y}_2) + 2n\epsilon_n. \tag{14}$$

Inequality (13) follows from the data processing inequality, while (14) is a consequence of the independence of $\mathbf{X}_1$ and $\mathbf{X}_2$.

At this point we state the following lemma. The proof, given in the Appendix, is essentially due to Körner and Marton [3].

Lemma: Let a discrete memoryless interference channel have inputs X_1, X_2 and outputs Y_1, Y_2. If $I(X_1;Y_1|X_2) \le I(X_1;Y_2|X_2)$ for all product probability distributions on $\mathcal{X}_1 \times \mathcal{X}_2$, then $I(\mathbf{X}_1;\mathbf{Y}_1|\mathbf{X}_2) \le I(\mathbf{X}_1;\mathbf{Y}_2|\mathbf{X}_2)$.

Applying this to (14), we obtain

$$n(R_1+R_2) \le I(\mathbf{X}_1;\mathbf{Y}_2|\mathbf{X}_2) + I(\mathbf{X}_2;\mathbf{Y}_2) + 2n\epsilon_n \tag{15}$$

$$= I(\mathbf{X}_1,\mathbf{X}_2;\mathbf{Y}_2) + 2n\epsilon_n \tag{16}$$

$$\le \sum_{i=1}^{n} I(X_{1i}, X_{2i}; Y_{2i}) + 2n\epsilon_n, \tag{17}$$

completing the proof of the converse.

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IV. Conclusion

The capacity region of the class of discrete interference channels with strong interference has been established. This class includes two classes of interference channels for which capacity regions were separately obtained. They are

a) channels with statistically equivalent outputs [2], [4], [5];
b) the class of channels with very strong interference, i.e., those for which $I(X_1;Y_1|X_2) \le I(X_1;Y_2)$ and $I(X_2;Y_2|X_1) \le I(X_2;Y_1)$ for all product probability distributions on the inputs [6], [7].

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Appendix
Proof of the Lemma

First we note that the hypothesis implies $I(X_1;Y_1|X_2,U) \le I(X_1;Y_2|X_2,U)$, where $U \to (X_1,X_2) \to (Y_1,Y_2)$ and $X_1 \to U \to X_2$ form Markov chains. Define $\boldsymbol{Y}^{n-1} = (Y_1, Y_2, \cdots, Y_{n-1})$. Then we have

$$\begin{aligned} I(\boldsymbol{X}_1;\boldsymbol{Y}_2|\boldsymbol{X}_2) - I(\boldsymbol{X}_1;\boldsymbol{Y}_1|\boldsymbol{X}_2) &= I\left(\boldsymbol{X}_1, \boldsymbol{Y}_2^{n-1}|\boldsymbol{X}_2\right) + I\left(\boldsymbol{X}_1; Y_{2n}|\boldsymbol{X}_2, \boldsymbol{Y}_2^{n-1}\right) \\ &\quad - I\left(\boldsymbol{X}_1; Y_{1n}|\boldsymbol{X}_2\right) - I\left(\boldsymbol{X}_1; \boldsymbol{Y}_1^{n-1}|\boldsymbol{X}_2, Y_{1n}\right) \\ &= I\left(\boldsymbol{X}_1, Y_{1n}; \boldsymbol{Y}_2^{n-1}|\boldsymbol{X}_2\right) + I\left(\boldsymbol{X}_1; Y_{2n}|\boldsymbol{X}_2, \boldsymbol{Y}_2^{n-1}\right) \\ &\quad - I\left(\boldsymbol{X}_1, \boldsymbol{Y}_2^{n-1}; Y_{1n}|\boldsymbol{X}_2\right) - I\left(\boldsymbol{X}_1; \boldsymbol{Y}_1^{n-1}|\boldsymbol{X}_2, Y_{1n}\right). \quad \text{(A1)} \end{aligned}$$

This follows from the fact that $Y_{1n} \to (\boldsymbol{X}_1, \boldsymbol{X}_2) \to \boldsymbol{Y}_2^{n-1}$ forms a Markov chain. Using the chain rule, we find

$$\begin{aligned} I(\boldsymbol{X}_1;\boldsymbol{Y}_2|\boldsymbol{X}_2) - I(\boldsymbol{X}_1;\boldsymbol{Y}_1|\boldsymbol{X}_2) &= I\left(Y_{1n}; \boldsymbol{Y}_2^{n-1}|\boldsymbol{X}_2\right) + I\left(\boldsymbol{X}_1^{n-1}; \boldsymbol{Y}_2^{n-1}|\boldsymbol{X}_2, Y_{1n}\right) \\ &\quad + I\left(X_{1n}; \boldsymbol{Y}_2^{n-1}|\boldsymbol{X}_2, Y_{1n}, \boldsymbol{X}_1^{n-1}\right) + I\left(X_{1n}; Y_{2n}|\boldsymbol{X}_2, \boldsymbol{Y}_2^{n-1}\right) \\ &\quad + I\left(\boldsymbol{X}_1^{n-1}; Y_{2n}|\boldsymbol{X}_2, \boldsymbol{Y}_2^{n-1}, X_{1n}\right) - I\left(\boldsymbol{Y}_2^{n-1}; Y_{1n}|\boldsymbol{X}_2\right) \\ &\quad - I\left(X_{1n}; Y_{1n}|\boldsymbol{X}_2, \boldsymbol{Y}_2^{n-1}\right) - I\left(\boldsymbol{X}_1^{n-1}; Y_{1n}|\boldsymbol{X}_2, \boldsymbol{Y}_2^{n-1}, X_{1n}\right) \\ &\quad - I\left(\boldsymbol{X}_1^{n-1}; \boldsymbol{Y}_1^{n-1}|\boldsymbol{X}_2, Y_{1n}\right) - I\left(X_{1n}; \boldsymbol{Y}_1^{n-1}|\boldsymbol{X}_2, Y_{1n}, \boldsymbol{X}_1^{n-1}\right). \quad \text{(A2)} \end{aligned}$$

The 3rd, 5th, 8th, and 10th terms of the right-hand side above are null, due to the memorylessness of the channel. Therefore,

$$\begin{aligned} I(\boldsymbol{X}_1;\boldsymbol{Y}_2|\boldsymbol{X}_2) - I(\boldsymbol{X}_1;\boldsymbol{Y}_1|\boldsymbol{X}_2) &= I\left(X_{1n}; Y_{2n}|\boldsymbol{X}_2, \boldsymbol{Y}_2^{n-1}\right) - I\left(X_{1n}; Y_{1n}|\boldsymbol{X}_2, \boldsymbol{Y}_2^{n-1}\right) \\ &\quad + I\left(\boldsymbol{X}_1^{n-1}; \boldsymbol{Y}_2^{n-1}|\boldsymbol{X}_2, Y_{1n}\right) - I\left(\boldsymbol{X}_1^{n-1}; \boldsymbol{Y}_1^{n-1}|\boldsymbol{X}_2, Y_{1n}\right). \quad \text{(A3)} \end{aligned}$$

Now, since

$$\left(X_{2n}, Y_{1n}\right) \to \left(\boldsymbol{X}_1^{n-1}, \boldsymbol{X}_2^{n-1}\right) \to \left(\boldsymbol{Y}_1^{n-1}, \boldsymbol{Y}_2^{n-1}\right),$$

$$\boldsymbol{X}_1^{n-1} \to \left(X_{2n}, Y_{1n}\right) \to \boldsymbol{X}_2^{n-1},$$

$$\left(\boldsymbol{X}_2^{n-1}, \boldsymbol{Y}_2^{n-1}\right) \to \left(X_{1n}, X_{2n}\right) \to \left(Y_{1n}, Y_{2n}\right),$$

and

$$X_{1n} \to \left(\boldsymbol{X}_2^{n-1}, \boldsymbol{Y}_2^{n-1}\right) \to X_{2n}$$

form Markov chains, it follows by induction that

$$I(\boldsymbol{X}_1;\boldsymbol{Y}_2|\boldsymbol{X}_2) - I(\boldsymbol{X}_1;\boldsymbol{Y}_1|\boldsymbol{X}_2) \ge 0. \quad \text{(A4)}$$

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Feedback Can at Most Double Gaussian Multiple Access Channel Capacity

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***Abstract* — The converse for the discrete memoryless multiple access channel is generalized and is used to derive strong bounds on the total capacity (sum of the rates of all the senders) of an m-user Gaussian multiple access channel in terms of the input covariance matrix. These bounds are used to show that the total capacity of the channel with feedback is less than twice the total capacity without feedback. The converse for the general multiple access channel is also used to show that for any m-user multiple access channel, feedback cannot increase the total capacity by more than a factor of m.**

I. Introduction

The simplest communication situation is when we have a single sender trying to send information to a single receiver. In many practical situations, however, we have two-way links—the receiver can also send back information to the sender (for example, telephone links). Although feedback is very common in practical channels, it is still only imperfectly understood and a large number of problems remain open on the capacity of channels with feedback. In this report, we establish bounds relating this capacity to the capacity without feedback for a class of multiple access channels. Our objective is to show that feedback cannot help very much in increasing the capacity of many practical channels.

The most important and rather surprising result in this area is due to Shannon [1], who established that feedback cannot in-

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