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ON DETERMINING THE IRRATIONALITY OF THE MEAN OF A RANDOM VARIABLE[1]

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A complexity approach is used to decide whether or not the mean of a sequence of independent identically distributed random variables lies in an arbitrary specified countable subset of the real line. A procedure is described that makes only a finite number of mistakes with probability one. This leads to some speculations on inference of the laws of physics and the computability of the physical constants.

1. Introduction and summary. Consider a sequence $x_1, x_2, \cdots$ of independent identically distributed coin tosses with unknown parameter $p = \Pr\{x_i = 1\}$. Let $S = \{r_i\}_1^\infty$ denote the rationals. Consider the countable set of hypotheses $H_i: p = r_i$, $i = 1, 2, \cdots$, together with the null hypothesis H_0: p is irrational. We wish to find a test which makes a decision after each new coin flip and makes only a finite number of mistakes with probability one for every $p \in [0, 1] - N_0$, where N_0 is a set of irrationals of Lebesgue measure zero.

It seems unlikely that there exists such a test, for several reasons. First, the obvious choice of the sample mean $\bar{x}_n = (1/n) \sum_{i=1}^n x_i$ as an estimate of p is always rational and seems to provide little basis for assuming p to be irrational. Second, although $|\bar{x}_n - p| \to 0$, any confidence interval centered at $\bar{x}_n$ contains an infinite number of rational and irrational parameters p which are likely causes for $x_1, x_2, \cdots, x_n$.

We shall return to the coin flipping problem after we have exhibited a proof of a somewhat more general result. Let $x_1, x_2 \cdots$ be a sequence of independent identically distributed random variables of unknown distribution with unknown mean $\mu = Ex$ and finite but unknown second moment. Let $S = \{\mu_1, \mu_2, \cdots\}$ be any countable subset of the real line R. For example, S could be the set of all algebraic numbers. The test will decide whether or not μ lies in S. Again, this test will make only a finite number of mistakes with probability one for any mean $\mu \notin N_0$, where N_0 is a subset of $R - S$ of Lebesgue measure zero. Thus, it is theoretically possible to determine whether empirically determined physical constants belong to certain sets of special numbers.

An outline of the decision procedure is as follows. At prescribed times $n(j)$, $j = 1, 2, \cdots$, an interval of width $2\delta_{n(j)}$ is centered about the sample mean $\bar{x}_{n(j)}$. Let $S = \{\mu_1, \mu_2, \cdots\}$ be an arbitrary but fixed enumeration of S,

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and let $i(\bar{x}_{n(j)}, \delta_{n(j)})$ denote the least index i such that μ_i lies in the interval $[\bar{x}_{n(j)} - \delta_{n(j)}, \bar{x}_{n(j)} + \delta_{n(j)}]$. Let $k_{n(j)}$ be an increasing sequence of positive integers. Then if

$$(1) \qquad i(\bar{x}_{n(j)}, \delta_{n(j)}) = i \leqq k_{n(j)}\,, \qquad \text{accept } H_i \text{ (i.e., decide } \mu = \mu_i\text{)}\,; \quad \text{otherwise accept } H_0 \text{ (i.e., decide } \mu \notin S\text{)}\,.$$

For times n, $n(j) \leqq n < n(j+1)$, continue to make the decision made at time $n(j)$. We shall prove that a proper choice of δ_n, k_n, $n(j)$ yields a test with the desired properties.

A similar result is obtained in Section 3 from a confidence interval specification of μ. The proof is somewhat simpler than that in the next section.

2. Choice of decision variables. Let $\mu = E\{X\}$ and $\sigma^2 = E(X - \mu)^2 < \infty$ denote the unknown mean and variance of x. Define the sample mean $\bar{x}_n = (1/n) \sum_1^n x_i$ and sample variance $s_n^2 = (1/n) \sum_1^n (x_i - \bar{x}_n)^2$. By the law of large numbers,

$$(2) \qquad \bar{x}_n \to \mu\,, \quad \text{w.p. } 1 \qquad \text{and}$$

$$(3) \qquad s_n^2 \to \sigma^2\,, \quad \text{w.p. } 1\,.$$

Although it is not essential for our arguments we shall also use the law of the iterated logarithm (see Chung [1]), which states that for $\varepsilon > 0$

$$(4) \qquad |\bar{x}_n - \mu| \leqq (1 + \varepsilon)(2\sigma^2 \log(\log n)/n)^{\frac{1}{2}} \quad \text{all but f.o. w.p. } 1,$$

i.e., with probability one this inequality is violated for only finitely many positive integers n.

Finally, we define

$$(5) \qquad i(t, \delta) = \min_i \{i : \mu_i \in [t - \delta, t + \delta]\}$$

to be the least index i such that μ_i lies in the interval $[t - \delta, t + \delta]$. (The calculation of $i(t, \delta)$ is easily seen to be finite for any effective enumeration of S (Minsky, [11], page 160) and any computable $(t, \delta) \in R \times R$.) We shall frequently use the property, following immediately from the definition, that

$$(6) \qquad 0 \leqq \delta \leqq \delta' \quad \text{implies} \quad i(t, \delta) \geqq i(t, \delta')\,.$$

If one interprets $i(\bar{x}_n, \delta)$ as the "complexity" of the explanation of $\bar{x}_n$ by S with accuracy δ, the test procedure has the following interpretation: Decide that unknown mean μ is given by the least complex explanation $\mu_{i(\bar{x}_n, \delta)}$, unless this explanation is too complex, in which case reject any explanation in S. This is Occam's razor, sharpened to admit the possibility of no explanation in S whatsoever.

To proceed, define the sequence of random variables $\{z_n\}$ and constant $\{a_n\}$ by

$$(7) \qquad z_n = (2s_n^2 \log(\log n)/n)^{\frac{1}{2}}$$

$$(8) \qquad a_n = (2\sigma^2 \log(\log n)/n)^{\frac{1}{2}}\,.$$

Equations (3) and (4) imply, for any $\varepsilon > 0$, that

$$(9) \qquad |\bar{x}_n - \mu| < (1 + \varepsilon)z_n \quad \text{all but f.o. w.p. 1.}$$

Now suppose that hypothesis H_i is true ($i \neq 0$), i.e., suppose $\mu = \mu_i$. Letting $\delta_n = (1 + \varepsilon)z_n$, it follows from (3), (7), (8) that $a_n < (1 + \varepsilon)z_n = \delta_n$ all but f.o. w.p. 1, thus implying that $\mu_i \varepsilon [\bar{x}_n - \delta_n, \bar{x}_n + \delta_n]$ all but f.o. w.p. 1, and therefore that

$$(10) \qquad i(\bar{x}_n, \delta_n) = i\,, \quad \text{all but f.o. w.p. 1.}$$

(Of course, if S is allowed to have repeated elements in its enumeration, $i(\bar{x}_n, \delta_n)$ will converge to the least index i such that $\mu = \mu_i$.) Consequently, if $k_n \to \infty$, and $\mu = \mu_i \in S$, then

$$(11) \qquad i(\bar{x}_n, \delta_n) = i \leqq k_n \quad \text{all but f.o. w.p. 1.}$$

Thus $i(\bar{x}_n, \delta_n)$ converges to the true index i and the test makes only a finite number of mistakes w.p. 1 if $\mu \in S$.

At this point we have a simple test among the countable hypotheses $\mu = \mu_i \in S$, but do not have a means for rejecting the hypothesis $\mu \in S$. This last step is less obvious.

In order to complete the description of the decision procedure, it remains to determine how k_n tends to infinity, and at what subsequence of times $n(j)$ the decision should be changed. These conditions will follow from investigation of the null hypothesis H_0: $\mu \notin S$.

Let us first observe that, for $\varepsilon > 0$,

$$(12) \qquad i(\bar{x}_n, (1 + \varepsilon)z_n) \geqq i(\mu, 2(1 + \varepsilon)z_n) \geqq i(\mu, 2(1 + \varepsilon)^2 a_n) \quad \text{all but f.o. w.p. 1,}$$

where the first inequality follows from (5) and (9) and the second inequality follows from (6) and $z_n < (1 + \varepsilon)a_n$ (all but f.o. w.p. 1).

We digress to observe that

$$(13) \qquad \lambda\{\mu \in R : i(\mu, \delta) \leqq k\} = \lambda\{\textstyle\bigcup_{i=1}^{k} [\mu_i - \delta, \mu_i + \delta]\} \leqq 2k\delta\,,$$

where λ denotes Lebesgue measure.

Let $n(j)$, $j = 1, 2, \cdots$ be a subsequence satisfying

$$(14) \qquad \textstyle\sum_{j=1}^{\infty} a_{n(j)} k_{n(j)} < \infty\,.$$

Then, by (13),

$$(15) \qquad \lambda\{\mu : i(\mu, 2(1 + \varepsilon)^2 a_{n(j)}) \leqq k_{n(j)}, \ \text{i.o.}\} \leqq 4(1 + \varepsilon)^2 \lim_{m\to\infty} \textstyle\sum_{j=m}^{\infty} a_{n(j)} k_{n(j)} = 0\,.$$

Let N_0 denote the null set of $R - S$ implied above for which $i(\mu, 2(1 + \varepsilon)^2 a_{n(j)}) \leqq k_{n(j)}$ infinitely often.

We can now conclude from (12) and (15) that, for $\mu \notin S \cup N_0$,

$$(16) \qquad i(\bar{x}_{n(j)}, \delta_{n(j)}) \geqq i(\mu, 2(1 + \varepsilon)^2 a_{n(j)}) \geqq k_{n(j)}\,, \quad \text{all but f.o. w.p. 1.}$$

Thus, under H_0: $\mu \notin S$, the decision procedure makes only a finite number of mistakes w.p. 1 for all $\mu \notin N_0$. Allowing the decisions to remain the same as that at time $n(j)$ for times n such that $n(j) \leqq n < n(j+1)$ results in a finite total number of mistakes w.p. 1.

Gathering these conditions together and recalling $s_n^2 = (1/n) \sum_{i=1}^n (x_i - \bar{x}_n)^2$, we have

THEOREM 1. *If $x_1, x_2, \cdots$, are independent identically distributed random variables with finite second moment, the decision procedure of Equation* (1), *in which $H_i(\mu = \mu_i)$ is accepted if $i(\bar{x}_{n(j)}, \delta_{n(j)}) = i \leqq k_{n(j)}$, and $H_0(\mu \notin S)$ is accepted if $i(\bar{x}_{n(j)}, \delta_{n(j)}) > k_{n(j)}$, where $\varepsilon > 0$, and δ_n, k_n, $n(j)$ satisfy*

$$k_n \to \infty , \qquad n(j) \nearrow \infty , \tag{17a}$$

$$\delta_n = (1+\varepsilon)(2 s_n^2 \log(\log n)/n)^{\frac{1}{2}} , \tag{17b}$$

$$\textstyle\sum_{j=1}^{\infty} k_{n(j)} (\log(\log n(j))/n(j))^{\frac{1}{2}} < \infty , \tag{17c}$$

will make only a finite number of mistakes with probability one in determining H_i: $\mu = \mu_i$, $i = 1, 2, \cdots$; H_0: $\mu \notin S = \{\mu_i : i = 1, 2, \cdots\}$ for any $\mu \notin N_0$, where N_0 is a null set of $R - S$.

COMMENTS. A possible choice of variables satisfying (17) is $n(j) = j^{6(1+\varepsilon)}$, $\varepsilon > 0$, and $k_{n(j)} = j$. Note also that the artifice of introducing a proper subsequence $n(j)$ is necessary, because setting $n(j) = j$ (allowing the decision to be changed after every observation) yields $\sum_{j=1}^{\infty} k_{n(j)} \delta_{n(j)} \geqq \sum_{j=1}^{\infty} \delta_j = \infty$, thus violating (17c). Apparently changing decisions too often may lead to an infinite number of errors. Finally, note that Theorem 1 tests an uncountable set of distributions against its uncountable complement.

If $x_1, x_2, \cdots$ are Bernoulli random variables with $\Pr\{x_i = 1\} = \mu = 1 - \Pr\{x_i = 0\}$, then $s_n^2 < \frac{1}{4}$, for all n. Thus the conditions simplify to

$$k_n \to \infty , \qquad n(j) \nearrow \infty , \tag{18a}$$

$$\delta_n = (1+\varepsilon)(\log(\log n)/2n)^{\frac{1}{2}} , \tag{18b}$$

$$\textstyle\sum_{j=1}^{\infty} k_{n(j)} (\log(\log n(j))/n(j)^{\frac{1}{2}} < \infty , \tag{18c}$$

and the complexity of the proof of the theorem can be somewhat reduced, yielding the following result.

COROLLARY 1. *If $x_1, x_2, \cdots$ are Bernoulli* rv'*s with unknown parameter μ, then the decision variables of* (18) *yield a decision procedure making only a finite number of mistakes with probability one in determining $\mu \in S$ vs. $\mu \notin S$ for $\mu \notin N_0$.*

When Corollary 1 was mentioned, D. Blackwell provided a beautiful application of a theorem of Doob [3] that also yields the result of this corollary. The idea is to put two finite weighting measures on the rationals and irrationals and compute the a posteriori probabilities of the hypotheses by Bayes' rule. By the Martingale theorem (Doob [3]), the a posteriori probability will converge to 1 on the correct hypothesis w.p. 1, and the result follows. Elegant as this approach is, it introduces weighting measures and the difficult attendant computations of the a posteriori probabilities. This Bayes approach also tends to obscure the "complexity" interpretation which seems to be the underlying idea. Apparently this Bayesian approach cannot be extended to prove Theorem 1 because of the difficulty of placing an interesting measure on the uncountable set of all distributions with a given mean with finite variance. See also the generalization that is achieved in Theorem 2 in Section 3, where a confidence interval point of view is taken.

Turning to a somewhat different method of revelation of a real number μ, suppose that a real number $\mu \in [0, 1]$ is revealed digit by digit. We can modify Theorem 1 (or use Theorem 2 of the next section directly) to obtain

COROLLARY 2. *If a real number $\mu = .\mu_1\mu_2\mu_3 \cdots \varepsilon[0, 1]$ is revealed digit by digit, we find, after defining $i_n = i(.\mu_1\mu_2 \cdots \mu_n 500, \cdots, \delta_n)$, that the procedure*

$$\text{Decide } \mu = s_i, \quad \text{if } i_n = i \leqq k_n;$$
$$\text{Decide } \mu \notin S, \quad \text{if } i_n > k_n,$$

where

(19a) $$k_n \to \infty,$$

(19b) $$\delta_n = (\tfrac{1}{2})10^{-n},$$

(19c) $$\textstyle\sum_{n=1}^{\infty} k_n \delta_n < \infty,$$

yields a sequence of decisions making only a finite number of mistakes with probability one in determining $\mu \in S$ vs. $\mu \notin S$ for $\mu \notin N_0$, N_0 a null set of $R - S$.

COMMENT. $k_n = g^n$ suffices.

As an example of this calculation, we have tested the irrationality of $\pi/10$, $e/10$, and $\frac{1}{7}$, where S is the set of rationals in the unit interval enumerated in the order $(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \cdots)$. Thus, for example, the index of $\frac{1}{5}$ is 9.

Throughout we shall use $k_n = 9^n$ as the decision threshold. Let i_n denote the index of the first rational in the enumeration $\{s_1, s_2, \cdots\}$ that agrees with μ in the first n digits.

For the number $e/10 = .27182818284 \cdots$ we have

n	$.\mu_1 \cdots \mu_n$	s_{i_n}	i_n	k_n	$i_n\delta_n$	Decision
1	.2	$\frac{1}{4}$	6	9	.6	Rational
5	.27182	109/401	79,911	59,049	.799	Irrational
9	.271828182	12,973/47,725	1.139×10^{10}	3.8×10^{8}	1.139	Irrational

For the number $\pi/10 = .31415926 \cdots$ we have

n	$.\mu_1 \cdots \mu_n$	s_{i_n}	i_n	k_n	$i_n\delta_n$	Decision
1	.3	$\frac{1}{3}$	4	9	·4	Rational
5	.31415	71/226	27,273	59,045	.252	Rational
9	.314159265	51,464/163,815	1.3×10^{11}	3.8×10^{8}	13.417	Irrational

For the number $\frac{1}{7} = .142857142 \cdots$ we have

n	$.\mu_1 \cdots \mu_n$	s_{i_n}	i_n	k_n	$i_n \delta_n$	Decision
1	.1	$\frac{1}{6}$	13	9	1.3	Irrational
5	.14285	$\frac{1}{7}$	18	59,049	.00018	Rational
9	.142857142	$\frac{1}{7}$	18	3.8×10^8	1.8×10^{-9}	Rational

A reasonable choice of decision variables would probably decide $e/10$ to be irrational after the 7th digit, $\pi/10$ to be irrational after the 8th digit and, $\frac{1}{7}$ to be rational after the 2nd digit. Except for the number $\frac{1}{7}$, we have not analyzed the behavior beyond the 9th digit. A more complete table can be obtained from the author.

3. Speculations and another theorem. There have been some recent interesting suggestions of formulas for various dimensionless physical constants. For example, Lenz [10], noted in 1951 that the ratio m_p/m_e of the mass of the proton to the mass of the electron is very closely approximated by $6\pi^5$. No theoretical justification was provided in his one line note. More recently, Good [9] and Wyler [14, 15] have seriously reiterated this conjecture and given some admittedly ad hoc theoretical justification involving the calculation of volumes of unit spheres in phase spaces of the appropriate dimension. The observed value of the ratio of the mass of the proton to the mass of the electron is $1836.109 \pm .011$. The conjectured value $6\pi^5$ equals $1836.118 \cdots$. This agrees within the experimental accuracy of one part in 10^5.

More bizarre perhaps is a conjecture by Wyler concerning the fine structure constant $\alpha = 2\pi e^2/hc$. This constant is dimensionless. It would be exceedingly interesting if α turned out to be a computable number, for this would yield a finite calculation of the charge of the electron in terms of the apparently independent physical constants h (Planck's constant) and c (the speed of light). So far, we have only empirical derivations of the fundamental physical constants. Wyler [15] conjectured in January 1971 that $\alpha^{-1} = (9/8\pi^4)(\pi^5/2^4 5!)^{\frac{1}{4}}$. This formula agrees with experiment up to the present experimental accuracy of one part in 10^6. As in the case of m_p/m_e, the calculation involves ratios of volumes of spheres.

These hypotheses are interesting, especially since they support the informal view of Einstein [5] and others that there is a simple relation among all of the dimensionless physical constants, i.e., none are arbitrarily specifiable, any more than the circumference of a circle can be independently specified given the radius.

In this section we wish to address the question of the degree of belief that should be attached to these hypotheses. (The interesting reexamination of Bode's law by Good [8] and Efron [4] illustrates the difficulties of arriving at a universally agreed upon degree of belief in a given hypothesis.)

A deviation from existing orthodoxy on statistical tests is that we nowhere consider a stopping rule,[3] largely because we envision the physicist taking action on the basis of his current knowledge while his research for the laws of nature continues. It is futile to stop this process and declare a given law fixed forever.

In the context of the problem below it will be shown that only a finite number of actions in the infinite sequence of actions will be inconsistent with the true state of nature. This will be true despite the lack of certainty at any finite time of the true state of nature.

This paper suggests an orderly approach to the formulation of conjectures, whereby a conjecture is to be acted upon as if true only if it is sufficiently simple with respect to the observational error in the phenomenon it describes. Otherwise, no conjecture is accepted. The principle that "the simplest explanation is best" is Occam's razor [7, 12]. The condition that the explanation not be too complex is a refinement.

Suppose one is concerned with a certain dimensionless physical constant α. Suppose also at time n that one is given an experimental guess α_n of the correct value of α together with a confidence interval δ_n, and a confidence $1 - p_n$ where $\Pr\{|\alpha_n - \alpha| > \delta_n\} \leqq p_n$. Of course, the better the experiment, the smaller may be δ_n for a given confidence $1 - p_n$.

We are concerned with the hypothesis that α belongs to a certain set S of special real numbers. In particular, let S be the set of all computable real numbers, i.e., all real numbers for which there exists a finite length computer program that will generate approximations of arbitrary prescribed accuracy.[4] Clearly S is countable, because the programs of finite length can be enumerated. Let $s_1, s_2, \cdots$ be an arbitrary but fixed enumeration of S. Let i_n denote the least index i such that $\alpha_n - \delta_n \leqq s_i \leqq \alpha_n + \delta_n$. We shall call i_n the "complexity" of the explanation in S of the estimate α_n with accuracy δ_n and confidence $1 - p_n$.

Let k_j be any increasing sequence of integers tending to infinity. Choose $n(j)$, $j = 1, 2, \cdots$, to be an increasing sequence of integers such that

$$\sum_{j=1}^{\infty} p_n(j) < \infty \tag{20a}$$

and

$$\sum_{j=1}^{\infty} k_j \delta_{n(j)} < \infty \,. \tag{20b}$$

Assuming that experiments of arbitrary accuracy can be performed, given sufficient time, is equivalent to assuming the existence of a subsequence $m(n)$ such that $(p_{m(n)}, \delta_{m(n)}) \rightarrow (0, 0)$. Under these conditions, k_j and $n(j)$ can be chosen to satisfy the constraints above.

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[3] Some nice results on the existence of stopping rules for testing $\mu = \mu_0$ vs. $\mu \neq u_0$ with power 1 have been obtained by Darling and Robbins [2].

[4] The idea that special attention should be paid to the computable real numbers arises naturally, since only the computable real numbers can be finitely described. Good [6, footnote page 55], for example, points out in the Bayesian context that it would be "quite rational to concentrate a finite amount of probability at every 'computable' value of $x \cdots$."

We shall use the following decision procedure: At times $n(j)$, $j = 1, 2, \cdots$ decide $\alpha = s_{i_{n(j)}}$, if $i_{n(j)} \leqq k_j$; otherwise decide $\alpha \notin S$. At time n, $n(j) < n < n(j+1)$, continue to make the decision made at time $n(j)$.

By using a Borel–Cantelli argument, we modify Theorem 1 to conclude that this test will make only a finite number of mistakes with probability one, for any real number α, subject only to the condition that α does not belong to a certain null set of the complement of S. Thus if α *is* a special number, we shall eventually determine its true value after a finite number of mistakes; and if α is not special (and not in the null set), we shall also make this decision forever after a finite number of decision errors.

Collecting these ideas we have the following theorem based on confidence intervals.

THEOREM 2. *Let α_n, $n = 1, 2, \cdots$ be a sequence of random variables (the joint distribution of which depends on the real number α), and let δ_n, p_n, $n = 1, 2, \cdots$, be sequences of real numbers such that* $\Pr\{|\alpha_n - \alpha| > \delta_n\} \leqq p_n$, $\forall_\alpha$ *in the real line R. Then if $p_n \to 0$, $\delta_n \to 0$, the above decision procedure will make only a finite number of errors with probability one in determining the correct hypothesis among $H_i: \alpha = s_i$; $i = 1, 2, \cdots$; $H_0: \alpha \notin S = \{s_i : i = 1, 2, \cdots\}$, for $\alpha \notin N_0 \subseteqq R - S$, where N_0 has Lebesgue measure zero.*

PROOF. Choose $n(j)$ to satisfy (20a, b). Suppose $\alpha \in S$. Let i be the least index such that $\alpha = s_i$. By the Borel–Cantelli lemma, $\sum_{j=1}^{\infty} p_{n(j)} < \infty$ implies $i_{n(j)} \leqq i$ all but finitely often with probability one. But $\delta_n \to 0$ implies $i_{n(j)} \geqq i$ all but f.o. w.p. 1. Thus $i_{n(j)} = i \leqq k_j$ all but f.o. w.p. 1, and the theorem is proved for $\alpha \in S$.

Now suppose $\alpha \notin S$. Let $i(t, \delta)$ denote the least index i such that $t - \delta \leqq s_i \leqq t + \delta$. We know, for all $\alpha \in R$, that $\Pr\{|\alpha_n - \alpha| > \delta_n\} \leqq p_n$, and thus that $|\alpha_{n(j)} - \alpha| \leqq \delta_{n(j)}$ all but f.o. w.p. 1. Hence, by the triangle inequality, $[\alpha_{n(j)} - \delta_{n(j)}, \alpha_{n(j)} + \delta_{n(j)}] \subseteqq [\alpha - 2\delta_{n(j)}, \alpha + 2\delta_{n(j)}]$ all but f.o. w.p. 1; which in turn implies $i(\alpha_{n(j)}\delta_{n(j)}) \geqq i(\alpha, 2\delta_{n(j)})$ all but f.o. w.p. 1.

Note that $\mu\{\alpha : i(\alpha, 2\delta_{n(j)}) \leqq k_j\} \leqq 4k_j\delta_{n(j)}$. Therefore $\sum k_j\delta_{n(j)} \leqq \infty$ implies $\mu\{\alpha : i(\alpha, 2\delta_{n(j)}) \leqq k_j, \text{ i.o.}\} = 0$, or, equivalently, $i(\alpha, 2\delta_{n(j)}) > k_j$, all but f.o. w.p. 1, a.e. α. Finally, if $\sum_{j=1}^{\infty} k_j\delta_{n(j)} < \infty$, then

$$i_{n(j)} = i(\alpha_{n(j)}, \delta_{n(j)}) \geqq i(\alpha, 2\delta_{n(j)}) \geqq k_j ,$$

all but f.o. w.p. 1, a.e. α. Hence the set of real numbers N_0 for which the decision "$\alpha \in S$" is made infinitely often has Lebesgue measure zero. Thus for $\alpha \notin S \cup N_0$, the correct decision "$\alpha \notin S$" is made all but finitely often, and the theorem is proved.

Two comments are necessary:

1. While it is true that this sequence of decisions will eventually be correct for all time, we will never have the luxury of knowing at what time we have made our last mistake. This is a characteristic of the problem and is not a

fault of the test. One has theories, and refinements of theories, and no guarantee that the process will ever stop, given the countable infinity of possible finite explanations and the uncountable infinity of possible infinite explanations. However, if there is a finite theory, and the accuracy of experiments grows without bound, then the proposed test will eventually decide on this theory. This convergence is by no means guaranteed by the present means of arriving at conclusions. I suspect that decisions are changed too often (i.e., $n(j)$ grows too slowly) and that an infinite number of incorrect decisions will result.

In order to make practical use of these considerations it is necessary to be able to calculate the complexity i_n in a finite amount of time. Actually, for S equal to the set of all computable real numbers, i_n is not a computable function. Thus the previous results are true for S, given i_n, but we cannot guarantee the finite calculation of i_n. Because of this we should be content with a somewhat smaller set S' of special numbers: for example, the set of all real numbers generated from the integers by primitive recursive operations [13]. Since the set of primitive recursive functions contains almost every known function, the set S' contains almost any number we can think of. In particular, S' contains all the rationals and algebraic numbers as well as $6\pi^5$ and $(9/8\pi^4)(\pi^5/2^4 5!)^{\frac{1}{4}}$. It can be shown that there exists an algorithm for S' which calculates i_n in finite time for any real α and any n.

Returning to Wyler's conjecture, we argue that $(9/8\pi^4)(\pi^5/2^4 5!)^{\frac{1}{4}}$ is an acceptable conjecture for α^{-1} if the index i_n of this formula in a list of all formulae physicists are likely to conjecture for this phenomenon is much less than 10^6. Implicit in this is that the confidence interval δ_n is such that $p_n \approx 0$ and that the experimental accuracy is one part in 10^6.

The rule of thumb that arises is that conjecture s_{i_n} is accepted if $i_n \delta_n \ll 1$. (Note that $2 i_n \delta_n$ is an upper bound on the Lebesgue measure of the set of real numbers that have δ_n-approximations with complexity less than or equal to i_n.) By embedding this one-shot decision in a sequence of decisions, it is clear that the desired objective of a finite number of mistakes is achievable.

Thus Wyler's conjecture should be rejected unless the complexity i_n of the formula is much less than 10^6. Although there is no universally agreed upon list of formulas, I think it is fair to say that a list chosen independently of the knowledge of the experimental value of the fine structure constant would not have an index i_n for Wyler's formula less than 10^6.

The theoretical physicist's burden in this problem is to show that $(9/8\pi^4)/(\pi^5/2^4 5!)^{\frac{1}{4}}$ is not as complex as it seems, by showing how it may be simply derived with the aid of "known" physical laws. In other words, we allow some juggling of the order of the list as a concession to common sense. A modification of Theorem 2 can be made to allow this. The burden on the experimentalist is to reduce the experimental error δ_n. This will allow formulas of higher complexities to be considered as explanations and will also eliminate incorrectly held theories of lower complexity.

Apparently Wyler's formula is too complex to be accepted on the basis of the current evidence. On the other hand, the apparent discreteness of mass and energy in the universe (and the consequent countability of mass points and energy levels) suggests that the laws of physics and all dimensionless physical constants, and indeed all biases of coins, etc., can be accommodated by a finite theory and are therefore computable real numbers and functions. If we ignore the physical problems of obtaining a sequence of observations of unbounded accuracy, this proposition can be tested.

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