Capacity Theorems for Relay Channels

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Relay Channel

- Discrete-memoryless relay channel [vM’71]

\[
p(y, y_1|x, x_1)
\]

- Goal: Reliably communicate message \( W \in [1, 2^{nR}] \) from \( X \) to \( Y \) with the help of relay node \((X_1, Y_1)\)

- Relay transmission:
  - Classical: \( x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1(i-1)}) \)
  - Without-delay: \( x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1i}) \)
Capacity

- Capacity $C$ is the supremum over achievable rates $R$
- An infinite-letter characterization of the capacity is given by
  \[ C = \lim_{k \to \infty} \sup_{p(x^k), \{f_i\}_{i=1}^k} \frac{1}{k} I(X^k; Y^k) \]
- Not considered a satisfactory answer — computationally intractable?
- A “computable” description of capacity is not known in general
- What we known:
  - Single-letter characterizations of capacity for some special classes
  - Single-letter upper and lower bounds on $C$
My Encounter with The Relay Channel

- Visited University of Hawaii in Spring 1976
  - ARPA Aloha packet radio project
- David Slepian introduced me to the problem
  - Send data directly and via a satellite
- Worked on it as part of my PhD thesis and with my first student M. Aref
- Little interest from information theory and communication community for years
- Renewed interest motivated by wireless networks — work with S. Zahedi, M. Mohseni, N. Hassanpour, and J. Mammen
Early Work 1997–1980

[CE’79 ] Cover, El Gamal, “Capacity theorems for the relay channel,” IT, 1979
- Cutset upper bound on $C$
- Block Markov coding (decode-and-forward)
- Capacity of degraded relay channels
- Capacity of relay channel with feedback
- Side-information coding (compress-and-forward)
- General lower bound on $C$—combining partial decode-and-forward and compress and forward

Recent Work 2002–Present


○ Compress-and-forward with time sharing
○ Capacity of FD-AWGN relay channels with linear relaying functions
○ Bounds on minimum energy-per-bit that differ by factor $< 1.7$


El Gamal, Hassanpour, “Capacity Theorems for the relay-without-delay channel,” Allerton, 2005

Outline

- Classical relay:
  - Cutset upper bound
  - Partial decode-and-forward
  - Compress-and-forward
  - Upper and lower bounds on capacity of AWGN relay channel

- Relay-without-delay:
  - New cutset bound
  - Instantaneous relaying
  - Lower bound and capacity results (see poster by N. Hassanpour)
Cutset Upper Bound

- Upper bound on capacity of classical relay [CE’79]:

\[ C \leq \max_{p(x,x_1)} \min \{ I(X, X_1; Y), I(X; Y, Y_1|X_1) \} \]

- Max-flow Min-cut interpretation:

  \[
  \begin{array}{c}
  X_1 \\
  \quad \\
  X \\
  \quad \\
  Y \\
  \quad \\
  X \\
  \quad \\
  Y
  \end{array}
  \]

  \[
  \begin{array}{c}
  Y_1 : X_1 \\
  \quad \\
  X \\
  \quad \\
  Y
  \end{array}
  \]

  Multiple access bound

  Broadcast bound

- Bound is tight for all classes where capacity is known

- Not known whether it is tight in general — *no known tighter bound*
Partial Decode-and-Forward

• Lower bound [CE’79]:
\[
C \geq \max_{p(u,x,x_1)} \min \{ I(X, X_1; Y), I(X; Y|X_1, U) + I(U; Y_1|X_1) \}
\]

• Transmission in \(B\) blocks. In block \(b\):

\[
\begin{array}{cccccc}
W_1 & W_2 & W_3 & \cdots & W_{B-1} & W_B \\
1 & 2 & 3 & \cdots & B-1 & B
\end{array}
\]

○ The relay and sender “coherently” send information \((X_1)\) to \(Y\)
○ The sender also superposes a new message \(W_b\) (thr’ \(X\))
○ The relay decodes part of \(W_b\) (represented by \(U\))
○ The receiver decodes \(X_1\), which helps it decode \(W_{b-1}\) (\(U\) then \(X\))

• Bound is tight for all cases where capacity is known:

○ Degraded: \((X \rightarrow (Y_1, X_1) \rightarrow Y)\). Set \(U = X\) [CE’79]
○ Semideterministic: \((Y_1 = g(X, X_1))\). Set \(U = Y_1\) [EA’82]
○ Relay channel with orthogonal components \((X = (X_D, X_R), p(y|x_D, x_1)p(y_1|x_R, x_1))\). Set \(U = X_R\) [EZ’05]
Compress-and-Forward

- The relay can help even without decoding any part of the message

- Compress-and-forward scheme [CE’79]:
  - Again transmission in $B$ blocks
  - Relay “quantizes” the received sequence in previous block and sends it to the receiver (a la Wyner-Ziv)

  $$\hat{Y}_1
  \begin{array}{c}
  \rightarrow \\
  Y_1 : X_1,
  \end{array}
  \begin{array}{c}
  \rightarrow \\
  W \rightarrow X \rightarrow Y \rightarrow \hat{W}
  \end{array}
  $$

  - Relay and the sender do not cooperate (interfere with each other)

  $$C \geq \max_{p(x)p(x_1)p(\hat{y}_1|x_1,x_1)} \min\{I(X; Y, \hat{Y}_1|X_1), I(X, X_1; Y) - I(Y_1; \hat{Y}_1|X, X_1)\}$$

- Except for some asymptotic results, not optimal for any known class

- Partial decode-and-forward and compress-and-forward can be combined (Theorem 7 of [CE’79])
Compress-and-Forward with Time-Sharing [EMZ’06]

- Compress-and-forward achievable rate is not convex
- Can be improved by time-sharing, which gives the bound

\[ C \geq \max \min \{ I(X; Y, \hat{Y}_1 | X_1, Q), I(X, X_1; Y | Q) - I(Y_1; \hat{Y}_1 | X, X_1, Q) \} \]

where the maximization is over \( p(q)p(x|q)p(x_1|q)p(\hat{y}_1|y_1, x_1, q) \)

- Example: FD-AWGN relay channel [EMZ’06]
Summary

- We don’t know the capacity of the DM relay channel in general
- We have upper and lower bounds that coincide in some cases:
  - Cutset upper bound. Tight for all cases where we know capacity. Is it tight in general?
  - Partial decode-and-forward lower bound. Tight for all classes where we know capacity
  - Compress-and-forward with time-sharing? Is it optimal for any class?
  - Partial decode-and-forward and compress-and-forward with time-sharing can be combined. Is it optimal for any class?
- Need new coding strategies and tighter upper bound
Consider the following AWGN relay channel:

\[ X \xrightarrow{a} Y_1 : X_1 \xrightarrow{b} Z \]

- \( a, b > 0 \) are relative channel gains
- \( Z_1 \sim \mathcal{N}(0, 1) \) is independent of \( Z \sim \mathcal{N}(0, 1) \)
- Average power constraints: \( E(X^2) \leq P \) and \( E(X_1^2) \leq P \)
- Capacity with average power constraints not known for any \( 0 < a, b < \infty \)
Cutset and Decode-and-Forward [EMZ’06]

\[
C \left( \frac{\left(b\sqrt{a^2-1}+\sqrt{a^2-b^2}\right)^2}{a^2} P \right), \text{ if } a^2 \geq 1 + b^2
\]

\[
C \left( \frac{\left(ab+\sqrt{1+a^2-b^2}\right)^2}{(1+a^2)} P \right), \text{ if } a^2 \geq b^2
\]

\[
C \left( \max\{1, a^2\} P \right), \text{ otherwise}
\]

\[
C \left( (1 + a^2) P \right), \text{ otherwise}
\]

\[
C(x) \triangleq \frac{1}{2} \log_2(1 + x)
\]

- Partial decode-and-forward reduces to decode-and-forward
- Bounds never coincide
Weak Channel to Relay \((a \leq 1)\)

- When \(a \leq 1\), decode-and-forward lower bound reduces to \(C(P)\), i.e., \textit{direct transmission} lower bound
  - Decoding at the relay is not beneficial since everything the relay can decode is already decoded by the receiver
- But, relay can still help by forwarding some information about it’s received sequence
- We consider two strategies:
  - Compress-and-forward
  - Linear relaying functions
Compress-and-Forward [EMZ’06]

- Achievable rate using compress-and-forward
  \[ R = \max_{p(x)p(x_1)p(\hat{y}_1|y_1,x_1)} \min \{ I(X, X_1; Y) - I(Y_1; \hat{Y}_1|X, X_1), I(X; Y, \hat{Y}_1|X_1) \}, \]

- The optimal choice of \( p(x)p(x_1)p(\hat{y}_1|y_1,x_1) \) is not known

- Assume \( X, X_1 \) and \( \hat{Y}_1 \) Gaussian:

\[ C \geq C \left( P \left( 1 + \frac{a^2 b^2 P}{P(1 + a^2 + b^2) + 1} \right) \right) \]

- As \( b \to \infty \), this bound becomes tight (coincides with broadcast bound)
Comparison of Bounds

- $a = 1$ and $b = 2$
Frequency-Division AWGN Relay Channel [EMZ’06]

• Model motivated by wireless communication:

\[ X \rightarrow a \rightarrow Y_1 : X_1 \rightarrow b \rightarrow Y_R \]

\[ X \rightarrow 1 \rightarrow Y_S \rightarrow Z_S \]

• Bounds on capacity of FD-AWGN relay

<table>
<thead>
<tr>
<th>Decode-and-forward</th>
<th>Cutset Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(P(1 + b^2 + b^2P)) ), if ( a^2 \geq 1 + b^2 + b^2P )</td>
<td>( C(P(1 + b^2 + b^2P)) ), if ( a^2 \geq b^2 + b^2P )</td>
</tr>
<tr>
<td>( C(\max{1, a^2}P) ), otherwise</td>
<td>( C((1 + a^2)P) ), otherwise</td>
</tr>
</tbody>
</table>

• If \( a^2 \geq 1 + b^2 + b^2P \) capacity is \( C\left( P \left( 1 + b^2 + b^2P \right) \right) \) (decode-forward)

• Model same as without delay
FD-AWGN – Weak Channel to Relay ($a \leq 1$) [EMZ’06]

- Again if $a \leq 1$, decode-and-forward reduces to direct transmission

- Can improve rate using compress-and-forward. Using Gaussian signals, we obtain:

  $$C \geq C \left( P \left( 1 + \frac{a^2 b^2 P(1 + P)}{a^2 P + (1 + P)(1 + b^2 P)} \right) \right)$$

- As $b$ or $P \to \infty$, compress-and-forward becomes optimal
FD-AWGN – Compress-and-Forward With Time-Sharing

- For small $P$, compress-and-forward is ineffective due to low SNR of $Y_1$

- Rate can be improved by time-sharing at the broadcast side ($X$ sends at power $P/\alpha$ for fraction $0 < \alpha \leq 1$ and 0, otherwise) to:

$$C \geq \max_{0 < \alpha \leq 1} \alpha C \left( \frac{P}{\alpha} \left( 1 + \frac{a^2}{1 + \left(1 + \frac{a^2 P}{P+\alpha}\right)} \right) \right)$$
Comparison of Bounds for FD AWGN

- Example: $a = 1$ and $b = 2$
Linear Relaying

• Assume relay can only transmit a linear combination of the past received symbols, i.e., \( x_{1(i+1)} = \sum_{j=1}^{i} d_{ij} y_{1j} \)

• Using vector notation

\[
X_1 = DY_1,
\]

where \( D = [d_{ij}]_{k \times k} \) is a strictly lower triangular matrix and

\[
X_1 = (X_{11}, X_{12}, \ldots, X_{1k})^T \quad \text{and} \quad Y_1 = (Y_{11}, Y_{12}, \ldots, Y_{1k})^T
\]

• Let \( C^{(l)} \) be the capacity with linear relaying subject to the power constraints
Capacity with Linear Relaying

- Capacity with linear relaying can be expressed as

\[ C^{(l)} = \sup_k \frac{1}{k} C_k^{(l)} = \lim_{k \to \infty} \frac{1}{k} C_k^{(l)} \]

where \( C_k^{(l)} = \sup_{P_{X^k,Y^k}} I(X^k; Y^k) \)

subject to

- Sender power constraint: \( \sum_{i=1}^{k} E(X_i^2) \leq kP \)
- Relay power constraint: \( \sum_{i=1}^{k} E(x_{1i}^2) \leq kP \)
- Causality constraint: \( D \) strictly lower triangular
Example [EMZ’04]

- Let $X_1 \sim \mathcal{N}(0, 2\alpha P)$, and $X_2 = \sqrt{\frac{1-\alpha}{\alpha}}X_1$ and $X_{12} = dY_{11}$ where $d$ is chosen to satisfy relay power constraints.

- The achievable rate is

$$R = \frac{1}{2}I(X_1, X_2; Y_1, Y_2)$$

$$= \max_{0 \leq \alpha \leq 1} \frac{1}{2} C \left( 2\alpha P \left( 1 + \frac{\left( \sqrt{(1 - \alpha)/\alpha + abd} \right)^2}{1 + b^2 d^2} \right) \right) \leq \frac{1}{2} C^{(l)}_2 \leq C^{(l)}$$
Comparison of Bounds

- Example: \( a = 1 \) and \( b = 2 \)
Capacity with Linear Relaying for General AWGN

- Gaussian distribution $P_{X^k}$ maximizes $C_k^{(l)} = \sup_{P_{X^k}; D} I(X^k; Y^k)$

- Problem reduces to:

  Maximize $\lim_{k \to \infty} \frac{1}{2k} \log \frac{|(I + abD)\Sigma_x(I + abD)^T + (I + b^2DD^T)|}{|(I + b^2DD^T)|}$

  Subject to $\Sigma_x \succeq 0$

  $\text{tr}(\Sigma_x) \leq kP$

  $\text{tr}(a^2\Sigma_x D^T D + D^T D) \leq kP$

  $d_{ij} = 0$ for $j \geq i$

- $\Sigma_x = E(XX^T)$ and $D$ are variables of the problem

- Sequence of non-convex problems (open problem)

- “Single-letter” characterization can be found for FD-AWGN relay channel
• Example: Consider the following “amplify-and-forward” scheme

\[
\begin{align*}
X & \sim \mathcal{N}(0, P) \quad \text{and} \quad X_1 = dY_1 \quad \text{where} \quad d \quad \text{is chosen to satisfy the relay power constraint}
\end{align*}
\]

• The achievable rate for this scheme is given by

\[
C_1^{(l)} = C \left( P \left( 1 + \frac{a^2 b^2 P}{1 + (a^2 + b^2)P} \right) \right)
\]
Example (continued)

- $C_1^{(l)}$ is convex for small $P$, concave for large $P$
- Rate can be improved with time-sharing on broadcast side

$$ C^{FD-L}(P) \geq \max_{0<\alpha,\theta\leq1} \bar{\alpha}C \left( \frac{\theta P}{\bar{\alpha}} \right) + \alpha C \left( \frac{\theta P}{\alpha} \left( 1 + \frac{a^2b^2P}{a^2\theta P + b^2P + \alpha} \right) \right) $$
Capacity of FD-AWGN Relay Channel with Linear Relaying

- Capacity with linear relaying for the FD-AWGN model can be expressed as

\[ C^{(l)} = \sup_k \frac{1}{k} C_k^{(l)} = \lim_{k \to \infty} \frac{1}{k} C_k^{(l)}, \]

where

\[ C_k^{(l)} = \sup_{P, X^k, D} I(X^k; Y_S^k, Y_R^k) \]

Subject to:

- Sender power constraint: \( \sum_{i=1}^{k} E(X_i^2) \leq kP \)
- Relay power constraint: \( \sum_{i=1}^{k} E(X_{1i}^2) \leq kP \)
- Causality constraint: \( D \) lower triangular
Optimization Problem

- Gaussian input distribution $P_{X^k}$ maximizes $C_k^{(l)} = \sup_{X^k, D} I(X^k, Y_S^k, Y_R^k)$

- Finding $C_k^{(l)}$ reduces to

Maximize

$$\frac{1}{2k} \log_2 \left| \begin{bmatrix}
I + \Sigma_x & ab\Sigma_x D^T \\
abD\Sigma_x & a^2b^2 D\Sigma_x D^T + (I + b^2 DD^T)
\end{bmatrix} \right|$$

Subject to

$\Sigma_x \succeq 0$

$\text{tr}(\Sigma_x) \leq kP$

$\text{tr}(a^2\Sigma_x D^T D + D^T D) \leq kP$, and $d_{ij} = 0$ for $j > i$

- $\Sigma_x = E(XX^T)$ and lower triangular $D$ are the optimization variables

- This is a non-convex problem in $(\Sigma_x, D)$ with $k^2 + k$ variables

- For a fixed $D$, problem is convex in $\Sigma_X$ (waterfilling optimal); however, finding $D$ for a fixed $\Sigma_X$ is a non-convex problem
Key Steps of the Proof

• Can show that diagonal \( \Sigma_x \) and \( D \) suffice

• Objective function reduces to

\[
\text{Maximize} \quad \frac{1}{2k} \log_2 \prod_{i=1}^{k} \left( 1 + \sigma_i \left( 1 + \frac{a^2b^2d_i^2}{1 + b^2d_i^2} \right) \right)
\]

Subject to \( \sigma_i \geq 0 \), for \( i = 1, 2, \ldots, k \), \( \sum_{i=1}^{k} \sigma_i \leq kP \), and

\[
\sum_{i=1}^{k} d_i^2 (1 + a^2 \sigma_i) \leq kP
\]

• This still is a non-convex optimization problem — time-share between \( k \) amplify-forward schemes

• At the optimum point:
  
  ◦ If \( \sigma_i = 0 \) it is easy to show that \( d_i = 0 \)
  
  ◦ If \( d_i = d_j = 0 \) (direct-transmission) then \( \sigma_i = \sigma_j \) (concavity of \( \log \) function)
Proof (continued)

- Problem reduces to:

\[
\text{Maximize } \frac{1}{2k} \log_2 \left( 1 + k \theta_0 P / k_0 \right)^{k_0} \prod_{i=k_0+1}^{k} \left( 1 + \sigma_i \left( 1 + a^2 b^2 d_i^2 / (1 + b^2 d_i^2) \right) \right) \\
\text{Subject to } d_i > 0, \text{ for } i = k_0 + 1, \ldots, k, \quad \sum_{i=k_0}^{k} \sigma_i \leq k \left( 1 - \theta_0 \right) P, \text{ and} \\
\sum_{i=k_0+1}^{k} d_i^2 \left( 1 + a^2 \sigma_i \right) \leq k P
\]

- By KKT conditions, at the optimum, there are \( \leq 4 \) non-zero \((\sigma_j, d_j)\)

\[
\text{Maximize } \frac{1}{2k} \log_2 \left( 1 + k \theta_0 P / k_0 \right)^{k_0} \prod_{j=1}^{4} \left( 1 + \sigma_j \left( 1 + a^2 b^2 d_j^2 / (1 + b^2 d_j^2) \right) \right)^{k_j} \\
\text{Subject to } d_j > 0, \text{ for } j = 1, \ldots, 4, \quad \sum_{j=1}^{4} k_j \sigma_j \leq k \left( 1 - \theta_0 \right) P, \text{ and} \\
\sum_{j=1}^{4} k_j d_j^2 \left( 1 + a^2 \sigma_j \right) \leq k P
\]
Taking the limit as $k \to \infty$, we obtain

$$C^{(l)} = \max \alpha_0 \mathcal{C} \left( \frac{\theta_0 P}{\alpha_0} \right) + \sum_{j=1}^{4} \alpha_j \mathcal{C} \left( \frac{\theta_j P}{\alpha_j} \left( 1 + \frac{a^2 b^2 \eta_j}{1 + b^2 \eta_j} \right) \right),$$

subject to $\alpha_j, \theta_j \geq 0$, $\eta_j > 0$, $\sum_{j=0}^{4} \alpha_j = \sum_{j=0}^{4} \theta_j = 1$, and

$$\sum_{j=1}^{4} \eta_j \left( a^2 \theta_j P + \alpha_j \right) = P$$

- Time-share between direct transmission and 4 amplify-forward regimes
- A non-convex optimization problem, but with only 14 variables and 3 constraints
- We haven’t been able to find any example where optimal requires $> 2$ amplify-forward regimes (in addition to direct transmission)
Comparison with Other Schemes

- Example: $a = 1$ and $b = 5$

- As $b \to \infty$, $C^{(l)}$ approaches the cutset upper bound
Summary

- For general AWGN relay channel:
  - Bounds are never tight for any $0 < a, b < \infty$
  - Compress-and-forward becomes optimal as $b \to \infty$
  - Linear relaying can beat compress-and-forward — we don’t have a computable expression for capacity with linear relaying

- For FD-AWGN relay channel:
  - Decode-and-forward is optimal (bound coincides with cutset) when $a^2 \geq 1 + b^2 + b^2P$
  - Compress-and-forward becomes optimal as $b$ or $P \to \infty$. Time-sharing improves rate for small $P$
  - We have a computable form of capacity with linear relaying — time-sharing between direct transmission and at most 4 amplify-and-forward regimes
Relay-Without-Delay (RWD)

- Suppose the delay from the sender $X$ to the receiver $Y$ is longer than to the relay $(X_1, Y_1)$, so that $x_{1i}$ can depend on its current, in addition to past received symbols, i.e.,

$$x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1(i-1)}, y_{1i})$$

- Relay-without-delay:

- Again, wish to reliably communicate $W \in [1, 2^{nR}]$ from $X$ to $Y$
Generalized Cutset Bound

- Upper bound on capacity of RWD channel:

\[
C \leq \max_{p(v,x), f(v,y_1)} \min \{ I(V, X; Y), I(X; Y, Y_1|V) \},
\]

where \( x_1 = f(v, y_1) \), and \(|V| \leq \min\{|Y|, |X| \cdot |X_1|\} + 1\)

- Proof follows similar lines to cutset bound:
  - Define \( V_i = Y_1^{i-1} \), so \( X_{1i} = f_i(V_i, Y_{1i}) \)
  - Key observation: \((W, Y^{i-1}) \rightarrow (X_i, V_i) \rightarrow (Y_i, Y_{1i})\)

- This bound can be strictly larger than the cutset bound

- Bound applies to classical case, but
  - \( x_1 \) is a function only of \( v \) and the bound reduces to the classical cutset bound
Proof

• Use standard Fano's inequality argument.

\[ nR \leq I(W; Y^n) + n\epsilon_n, \text{ where } \epsilon_n \to 0 \]

• Consider

\[
I(W; Y^n) \leq \sum_{i=1}^{n} I(W; Y_i|Y^{i-1}) \\
\leq \sum_{i=1}^{n} (H(Y_i) - H(Y_i|W, Y^{i-1})) \\
\leq \sum_{i=1}^{n} (H(Y_i) - H(Y_i|W, Y^{i-1}, Y^{i-1}, X_i)) \\
= \sum_{i=1}^{n} (H(Y_i) - H(Y_i|Y^{i-1}, X_i)) \\
= \sum_{i=1}^{n} I(X_i, V_i; Y_i) = nI(X_Q, V_Q; Y_Q|Q) \leq nI(X, V; Y)
\]
Next consider

\[ I(W; Y^n) \leq I(W; Y^n, Y^n_1) \]

\[ = \sum_{i=1}^{n} I(W; Y_i, Y_1i|Y_i^{i-1}, Y_1^{i-1}) \]

\[ = \sum_{i=1}^{n} H(Y_i, Y_1i|Y_i^{i-1}, Y_1^{i-1}) - H(Y_i, Y_1i|Y_i^{i-1}, Y_1^{i-1}, W) \]

\[ \leq \sum_{i=1}^{n} H(Y_i, Y_1i|V_i) - H(Y_i, Y_1i|Y_i^{i-1}, Y_1^{i-1}, W, X_i) \]

\[ \leq \sum_{i=1}^{n} H(Y_i, Y_1i|V_i) - H(Y_i, Y_1i|V_i, X_i) \]

\[ = \sum_{i=1}^{n} I(X_i; Y_i, Y_1i|V_i) = nI(X; Y, Y_1|V) \]
Instantaneous Relaying

- Any lower bound on the capacity of the classical relay channel, e.g., decode-and-forward, is a lower bound on the RWD channel.
- Instantaneous relaying: Ignore the past and set $X_1 = f(Y_1)$.

\[ R = \max_{p(x), f(y_1)} I(X; Y) \]

- This strategy can be optimal and can achieve higher rate than classical cutset bound!
Consider the following DW relay channel [Sato’76]. $Y_1 = X$ and $X_1 = 0$ or $X_1 = 1$.

- The capacity of the classical case is $1.161878$ bits/transmission [CE’79]
  - Coincides with cutset bound
- The upper bound on the capacity without delay gives:
  $$C \leq 2(\log 3 - 1) = 1.169925 \text{ bits/transmission}$$
- Instantaneous relaying with input distribution $(\frac{3}{9}, \frac{2}{9}, \frac{4}{9})$ and the mapping from $X$ to $X_1$ of $0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 1$ achieves this bound!!
- Capacity of this RWD channel $> \text{classical, which is } = \text{ cutset bound}$
• Same model as before (but without relay coding delay)

\[ Z_1 \sim \mathcal{N}(0, 1) \text{ is independent of } Z \sim \mathcal{N}(0, 1) \]

\[ x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1i}) \]

\[ E(X_2^2) \leq P \text{ and } E(X_1^2) \leq P \]

Recall that capacity of the classical case is not known for any \( 0 < a, b < \infty \)
Amplify-and-Forward

- Consider the following special case of instantaneous relaying:

- If $a \leq b$, the upper bound gives

  $$ C \leq C \left( (1 + a^2)P \right) $$

- Consider the case where $a^2 \leq b^2 \min \{1, P/(a^2P + 1)\}$

  Amplify-and-forward also gives

  $$ C \geq \max_{p(x), E(X^2) \leq P} I(X; Y) = C \left( (1 + a^2)P \right), $$

- Capacity in this case coincides with classical cutset bound

- Can capacity of AWGN RWD channel be $> \text{cutset bound}$?
New Lower Bound on Capacity of RWD Channel

- Idea: Use a superposition of partial decode-and-forward and instantaneous relaying
- Achieves the lower bound:

\[ C \geq \max_{p(u,v,x), f(v,y_1)} \min\{I(V, X; Y), I(U; Y_1|V) + I(X; Y|V, U)\} \]

- \(U\) represents the information decoded by the relay in partial decode-and-forward and \(V\) (replacing \(X_1\)) represents the information sent coherently by the sender and the relay to help the receiver decode the previous \(U\)
- Each \(x_{1i}^n\) codeword in partial decode-and-forward is replaced by a \(v^n \) codeword, and at time \(i\) the relay sends \(x_{1i} = f(v_i, y_{1i})\)

- This bound coincides with the generalized cutset bound for degraded and semi-deterministic RWD channels
- Same superposition idea can be used to obtain new compress-and-forward lower bound
Achievable Rate for AWGN RWD Channel

- Restrict previous scheme to superposition of decode-and-forward and amplify-and-forward:
  - Let $U = X = V + X'$, where $V \sim \mathcal{N}(0, \alpha P)$ and $X' \sim \mathcal{N}(0, \bar{\alpha} P)$ are independent, for $0 \leq \alpha \leq 1$ and $\bar{\alpha} = 1 - \alpha$
  - Let $X_1$ be a normalized convex combination of $Y_1$ and $V$
    $$X_1 = h(\beta Y_1 + \bar{\beta} V), \ 0 \leq \beta \leq 1$$

- Using the power constraint on the relay sender, we obtain
  $$h^2 = \frac{P}{(\bar{\beta} + a\beta)^2\alpha P + \beta^2(a^2\bar{\alpha} P + N)}$$

- Substituting the above choice of $(U, V, X, X_1)$, we obtain
  $$C_{RWD-AWGN} \geq \max_{\alpha, \beta} \min \left\{ C(a^2\bar{\alpha} P), C\left(\frac{\alpha P(bh(a\beta + \bar{\beta}) + 1)^2 + \bar{\alpha} P(bha\beta + 1)^2}{(1 + (\beta bh)^2)}\right)\right\}$$
Achievable Rate vs. Cutset vs. Upper Bound

Assume $a = 2$ and $b = 1$
# Classical vs. RWD Relay Channels

<table>
<thead>
<tr>
<th>Result</th>
<th>Classical Relay</th>
<th>RWD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacity</td>
<td>Not known in general</td>
<td>Not known in general</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Can be &gt; classical</td>
</tr>
<tr>
<td>Upper bound</td>
<td>$\max_{p(x, x_1)} \min {I(X, X_1; Y), I(X; Y, Y_1</td>
<td>X_1)}$</td>
</tr>
<tr>
<td></td>
<td>Cutset bound</td>
<td>Can be &gt; cutset bound</td>
</tr>
<tr>
<td>Degraded capacity $X \rightarrow (X_1, Y_1) \rightarrow Y$</td>
<td>$\max_{p(x, x_1)} \min {I(X, X_1; Y), I(X; Y_1</td>
<td>X_1)}$</td>
</tr>
<tr>
<td></td>
<td>Decode-forward</td>
<td>Decode-forward + instantaneous coding</td>
</tr>
<tr>
<td>Semi-det capacity $Y_1 = g(X)$</td>
<td>$\max_{p(x, x_1)} \min {I(X, X_1; Y), I(X; Y, Y_1</td>
<td>X_1)}$</td>
</tr>
<tr>
<td></td>
<td>Partial decode-forward + instantaneous coding</td>
<td></td>
</tr>
<tr>
<td>AWGN</td>
<td>Capacity not known for any $a, b \neq 0$</td>
<td>Capacity known for $a^2 \leq b^2 \min {1, \frac{P}{a^2 P + N}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Amplify-forward</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Can be in general &gt; classical</td>
</tr>
</tbody>
</table>
Conclusion

• Overview of known upper and lower bounds on relay and RWD channels
  ◦ Bounds are tight only is special cases
  ◦ We know as much about the RWD as classical relay, main difference is instantaneous relaying, but it is not sufficient

• Generalized cutset bound needed for RWD channel
  ◦ Generalization to relay networks with delays (see poster with J. Mammen)