Capacity Theorems for Relay Channels

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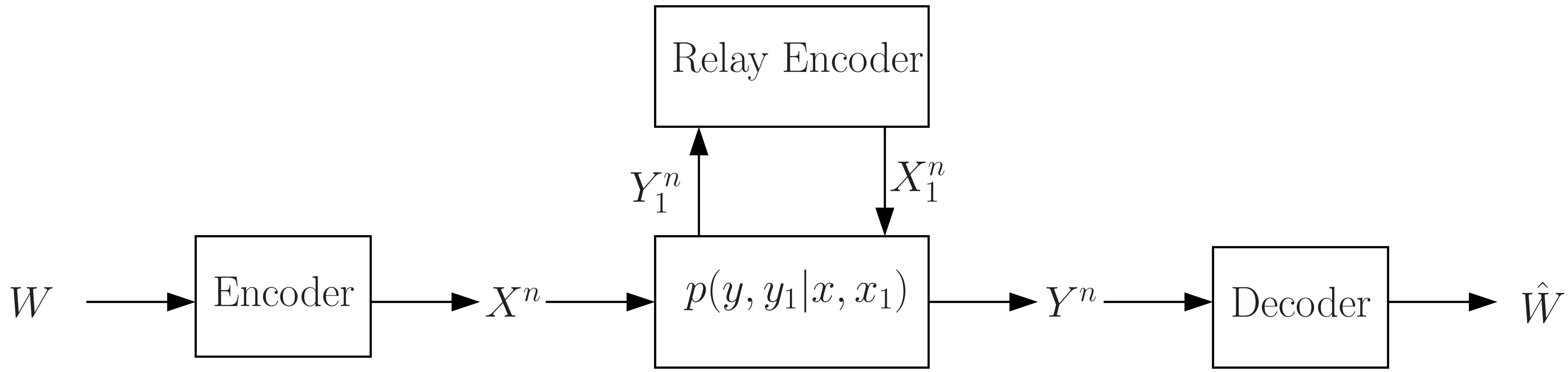
Relay Channel

- Discrete-memoryless relay channel [vM'71]



- Goal: Reliably communicate message $W \in [1, 2^{nR}]$ from X to Y with the help of relay node (X_1, Y_1)
- Relay transmission:
  - Classical: $x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1(i-1)})$
  - Without-delay: $x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1i})$

Capacity

- Capacity C is the supremum over achievable rates R
- An infinite-letter characterization of the capacity is given by

$$C = \lim_{k \to \infty} \sup_{p(x^k), \{f_i\}_{i=1}^k} \frac{1}{k} I(X^k; Y^k)$$

- Not considered a satisfactory answer — computationally intractable?
- A "computable" description of capacity is not known in general
- What we known:
    - Single-letter charcterizations of capacity for some special classes
    - Single-letter upper and lower bounds on C

My Encounter with The Relay Channel

- Visited University of Hawaii in Spring 1976
  - ARPA Aloha packet radio project
- David Slepian introduced me to the problem
  - Send data directly and via a satellite
- Worked on it as part of my PhD thesis and with my first student M. Aref
- Little interest from information theory and communication community for years
- Renewed interest motivated by wireless networks — work with S. Zahedi, M. Mohseni, N. Hassanpour, and J. Mammen

Early Work 1997–1980

[CE'79] Cover, El Gamal, "Capacity theorems for the relay channel," IT, 1979

- ◦ Cutset upper bound on C
- ◦ Block Markov coding (decode-and-forward)
- ◦ Capacity of degraded relay channels
- ◦ Capacity of relay channel with feedback
- ◦ Side-information coding (compress-and-forward)
- ◦ General lower bound on C—combining partial decode-and-forward and compress and forward

[EA'82] El Gamal, Aref, "The Capacity of the semi-determininstic relay channel,"IT Trans., 1982 (partial decode-and-forward)

- El Gamal, "On information flow in relay networks,"IEEE NTC, 1980 (cutset bound for relay networks)
- Aref, "Information flow in relay networks," PhD Thesis, Stanford 1980 (capacity of degraded relay networks)

Recent Work 2002–Present

[EZ'05] El Gamal, Zahedi, "Capacity of a class of of relay channels with orthogonal components," IT Trans., 2005 (partial decode-and-forward)

[EMZ'06] El Gamal, Mohseni, Zahedi, "Bounds on Capacity and Minimum Energy-Per-Bit for AWGN Relay Channels," to appear in IT Trans.

- Compress-and-forward with time sharing
- Capacity of FD-AWGN relay channels with linear relaying functions
- Bounds on minimum energy-per-bit that differ by factor < 1.7

[EH'05] El Gamal, Hassanpour, "Relay-without-delay," ISIT, 2005

El Gamal, Hassanpour, "Capacity Theorems for the relay-without-delay channel," Allerton, 2005

[EM'06] El Gamal, Mammen, "Relay networks with delays," UCSD, 2006

Outline

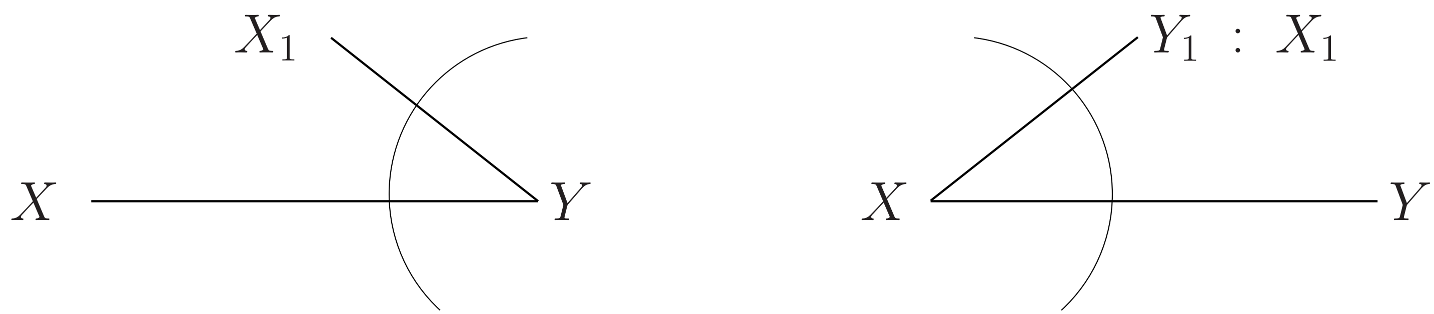
- Classical relay:
  - Cutset upper bound
  - Partial decode-and-forward
  - Compress-and-forward
  - Upper and lower bounds on capacity of AWGN relay channel
- Relay-without-delay:
  - New cutset bound
  - Instantaneous relaying
  - Lower bound and capacity results (see poster by N. Hassanpour)

Cutset Upper Bound

- Upper bound on capacity of classical relay [CE'79]:

$$C \leq \max_{p(x,x_1)} \min \{I(X, X_1; Y), I(X; Y, Y_1 | X_1)\}$$

- Max-flow Min-cut interpretation:



Multiple access bound   Broadcast bound

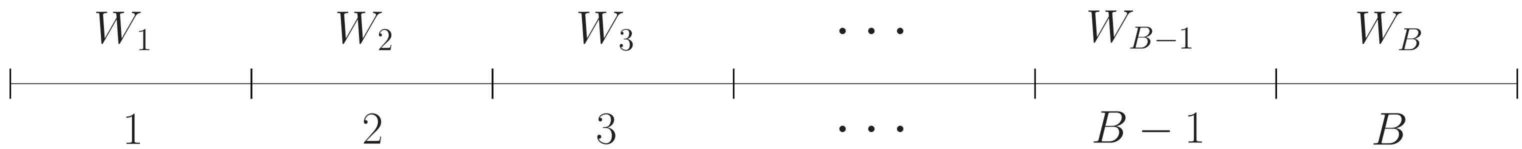
- Bound is tight for all classes where capacity is known
- Not known whether it is tight in general — *no known tighter bound*

Partial Decode-and-Forward

- Lower bound [CE'79]:

$$C \geq \max_{p(u,x,x_1)} \min \{ I(X, X_1; Y), I(X; Y | X_1, U) + I(U; Y_1 | X_1) \}$$

- Transmission in B blocks. In block b:



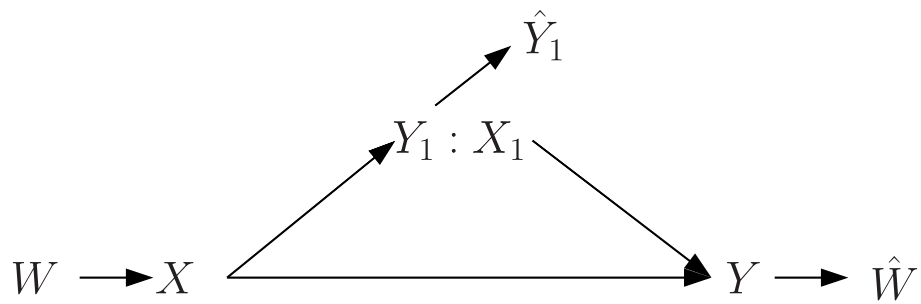
  - The relay and sender "coherently" send information (X_1) to Y
  - The sender also superposes a new message W_b (thr' X)
  - The relay decodes part of W_b (represented by U)
  - The receiver decodes X_1, which helps it decode W_{b-1} (U then X)

- Bound is tight for all cases where capacity is known:

  - Degraded: ($X \to (Y_1, X_1) \to Y$). Set $U = X$ [CE'79]
  - Semideterministic: ($Y_1 = g(X, X_1)$). Set $U = Y_1$ [EA'82]
  - Relay channel with orthogonal components ($X = (X_D, X_R)$, $p(y|x_D, x_1)p(y_1|x_R, x_1)$). Set $U = X_R$ [EZ'05]

Compress-and-Forward

- The relay can help even without decoding any part of the message
- Compress-and-forward scheme [CE'79]:
  - Again transmission in B blocks
  - Relay "quantizes" the received sequence in previous block and sends it to the receiver (a la Wyner-Ziv)



  - Relay and the sender do not cooperate (interfere with each other)

$$C \geq \max_{p(x)p(x_1)p(\hat{y}_1|y_1,x_1)} \min\{I(X;Y,\hat{Y}_1|X_1), I(X,X_1;Y) - I(Y_1;\hat{Y}_1|X,X_1)\}$$

- Except for some asymptotic results, not optimal for any known class
- Partial decode-and-forward and compress-and-forward can be combined (Theorem 7 of [CE'79])

Compress-and-Forward with Time-Sharing [EMZ'06]

- Compress-and-forward achievable rate is not convex
- Can be improved by time-sharing, which gives the bound

$$C \geq \max \min\{I(X;Y,\hat{Y}_1|X_1,Q), I(X,X_1;Y|Q) - I(Y_1;\hat{Y}_1|X,X_1,Q)\},$$

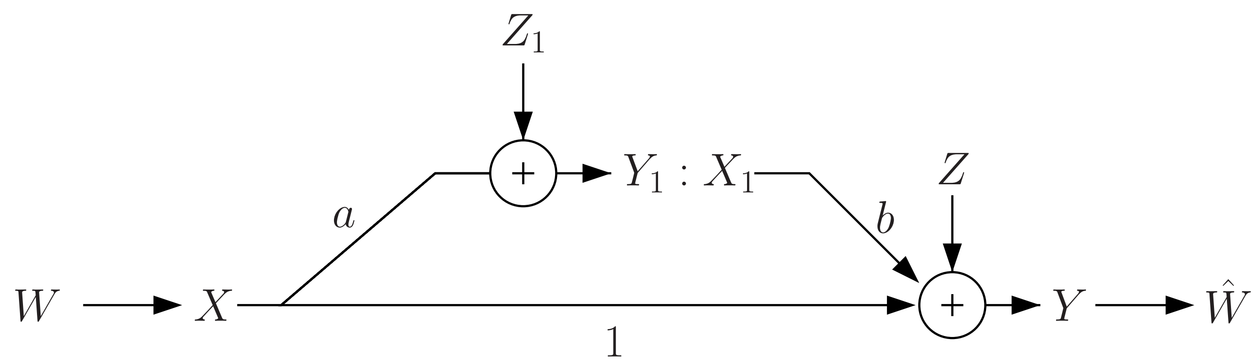
  where the maximization is over $p(q)p(x|q)p(x_1|q)p(\hat{y}_1|y_1,x_1,q)$
- Example: FD-AWGN relay channel [EMZ'06]

Summary

- We don't know the capacity of the DM relay channel in general
- We have upper and lower bounds that coincide in some cases:
  - Cutset upper bound. Tight for all cases where we know capacity. Is it tight in general?
  - Partial decode-and-forward lower bound. Tight for all classes where we know capacity
  - Compress-and-forward with time-sharing? Is it optimal for any class?
  - Partial decode-and-forward and compress-and-forward with time-sharing can be combined. Is it optimal for any class?
- Neeed new coding strategies and tighter upper bound

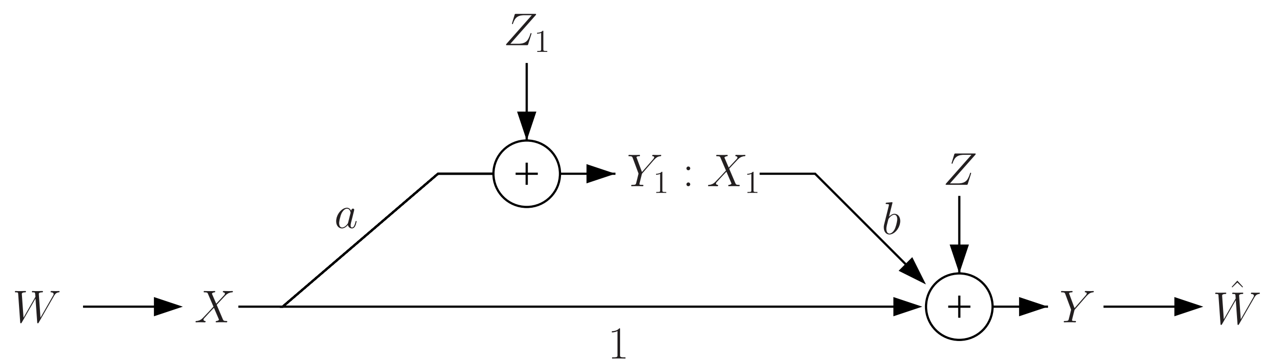
General AWGN Relay Channel

- Consider the following AWGN relay channel



- $a, b > 0$ are relative channel gains
- $Z_1 \sim \mathcal{N}(0, 1)$ is independent of $Z \sim \mathcal{N}(0, 1)$
- Average power constraints: $E(X^2) \leq P$ and $E(X_1^2) \leq P$
- Capacity with average power constraints not known for any $0 < a, b < \infty$

Cutset and Decode-and-Forward [EMZ'06]

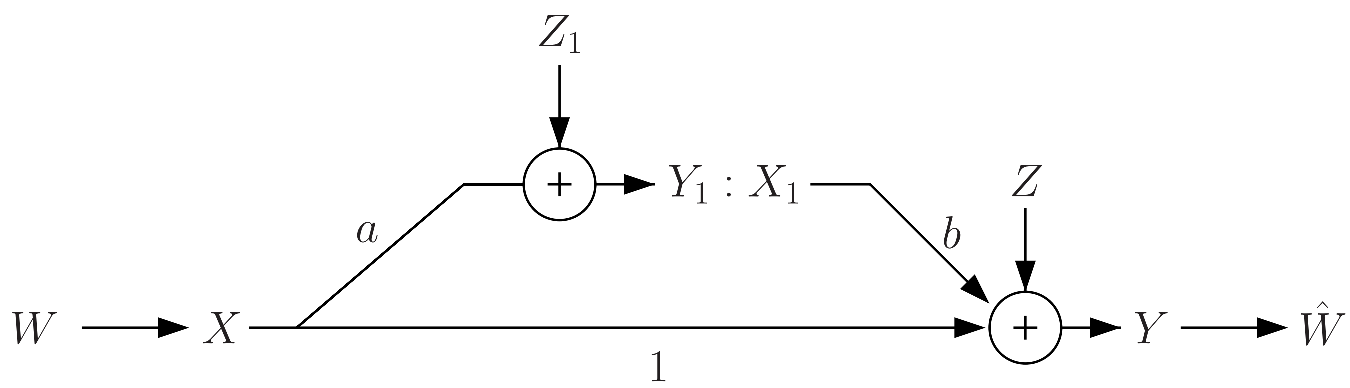


Decode-and-forward		Cutset Upper Bound	
$\mathcal{C}\left(\frac{\left(b\sqrt{a^2-1}+\sqrt{a^2-b^2}\right)^2 P}{a^2}\right)$,	if $a^2 \geq 1+b^2$	$\mathcal{C}\left(\frac{\left(ab+\sqrt{1+a^2-b^2}\right)^2 P}{(1+a^2)}\right)$,	if $a^2 \geq b^2$
$\mathcal{C}\left(\max\{1, a^2\}P\right)$,	otherwise	$\mathcal{C}\left((1+a^2)P\right)$,	otherwise

$\mathcal{C}(x) \stackrel{\triangle}{=} \frac{1}{2}\log_2(1+x)$

- Partial decode-and-forward reduces to decode-and-forward
- Bounds never coincide

Weak Channel to Relay ($a \leq 1$)



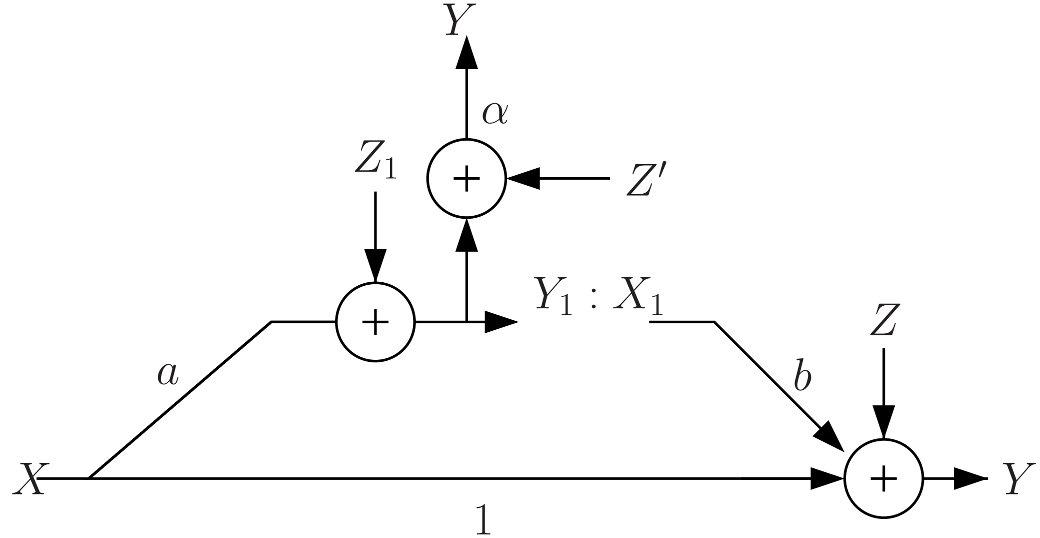
- When $a \leq 1$, decode-and-forward lower bound reduces to $\mathcal{C}(P)$, i.e., *direct transmission* lower bound
  - Decoding at the relay is not beneficial since everything the relay can decode is already decoded by the receiver
- But, relay can still help by forwarding some information about it's received sequence
- We consider two strategies:
  - Compress-and-forward
  - Linear relaying functions

Compress-and-Forward [EMZ'06]

- Achievable rate using compress-and-forward

$$R = \max_{p(x)p(x_1)p(\hat{y}_1|y_1,x_1)} \min\{I(X, X_1; Y) - I(Y_1; \hat{Y}_1|X, X_1), I(X; Y, \hat{Y}_1|X_1)\},$$

- The optimal choice of $p(x)p(x_1)p(\hat{y}_1|y_1, x_1)$ is not known
- Assume X, X_1 and $\hat{Y}_1$ Gaussian:

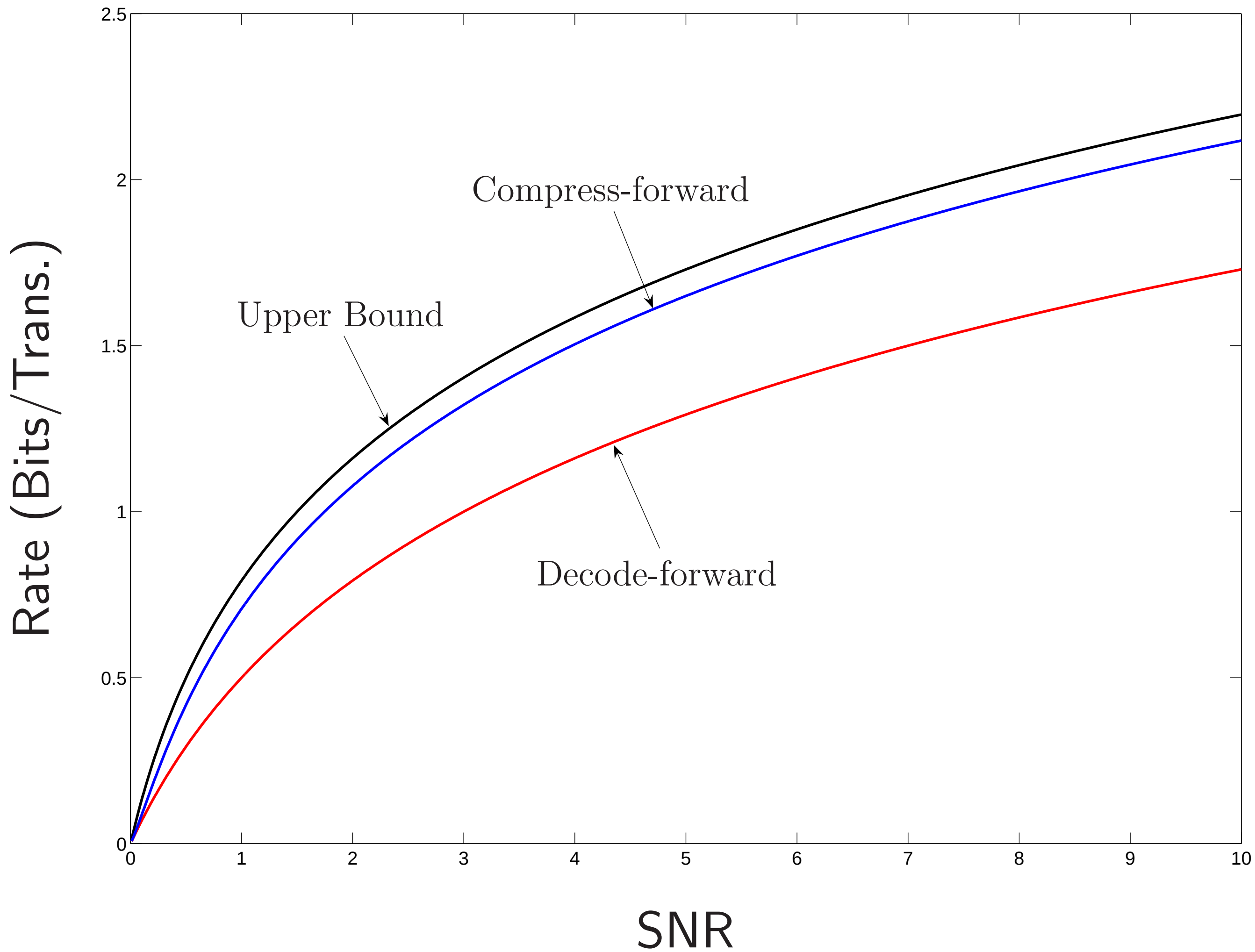


$$C \geq \mathcal{C}\left(P\left(1 + \frac{a^2b^2P}{P(1 + a^2 + b^2) + 1}\right)\right)$$

- As $b \to \infty$, this bound becomes tight (coincides with broadcast bound)

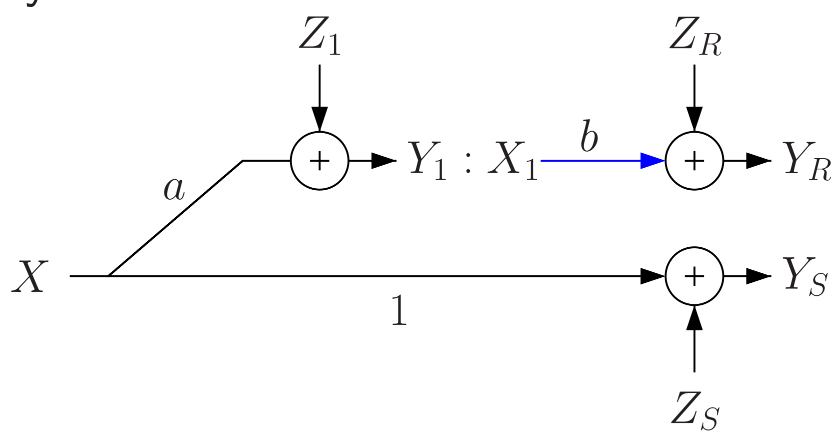
Comparison of Bounds

- $a = 1$ and $b = 2$



Frequency-Division AWGN Relay Channel [EMZ'06]

- Model motivated by wireless communication:



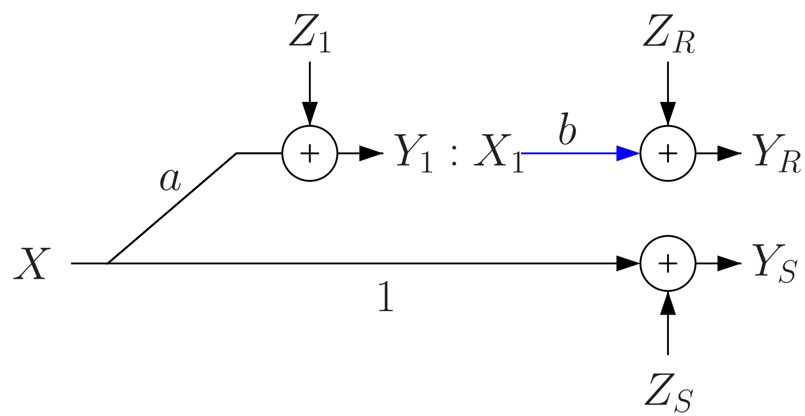
- Bounds on capacity of FD-AWGN relay

Decode-and-forward		Cutset Upper Bound	
$\mathcal{C}\left(P\left(1+b^2+b^2P\right)\right)$,	if $a^2 \geq 1+b^2+b^2P$	$\mathcal{C}\left(P\left(1+b^2+b^2P\right)\right)$,	if $a^2 \geq b^2+b^2P$
$\mathcal{C}\left(\max\{1,a^2\}P\right)$,	otherwise	$\mathcal{C}\left((1+a^2)P\right)$,	otherwise

- If $a^2 \geq 1+b^2+b^2P$ capacity is $\mathcal{C}\left(P\left(1+b^2+b^2P\right)\right)$ (decode-forward)
- Model same as without delay

FD-AWGN – Weak Channel to Relay $(a \le 1)$ [EMZ'06]

- Again if $a \le 1$, decode-and-forward reduces to direct transmission



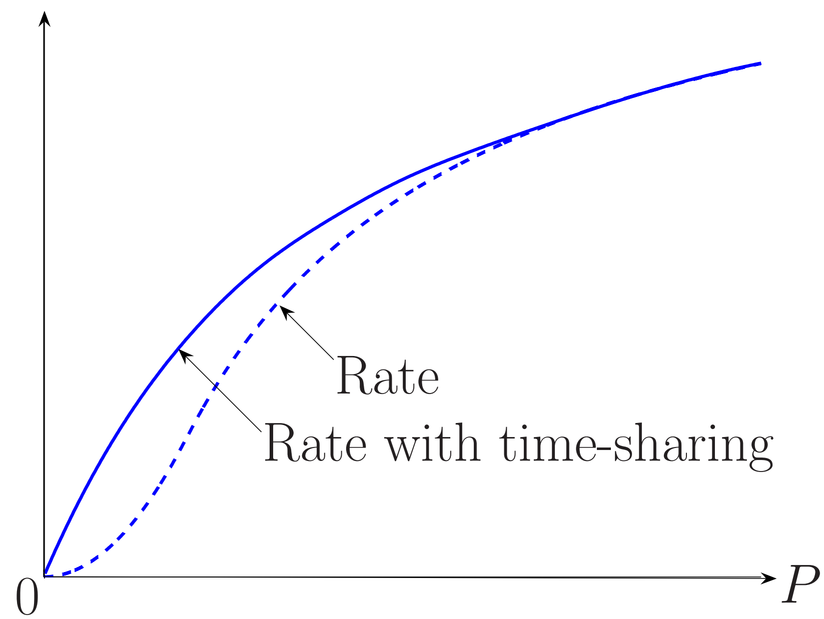
- Can improve rate using compress-and-forward. Using Gaussian signals, we obtain:

$$C \ge \mathcal{C}\left(P\left(1 + \frac{a^2b^2P(1+P)}{a^2P + (1+P)(1+b^2P)}\right)\right)$$

- As b or $P \to \infty$, compress-and-forward becomes optimal

FD-AWGN – Compress-and-Forward With Time-Sharing

- For small P, compress-and-forward is ineffective due to low SNR of Y_1

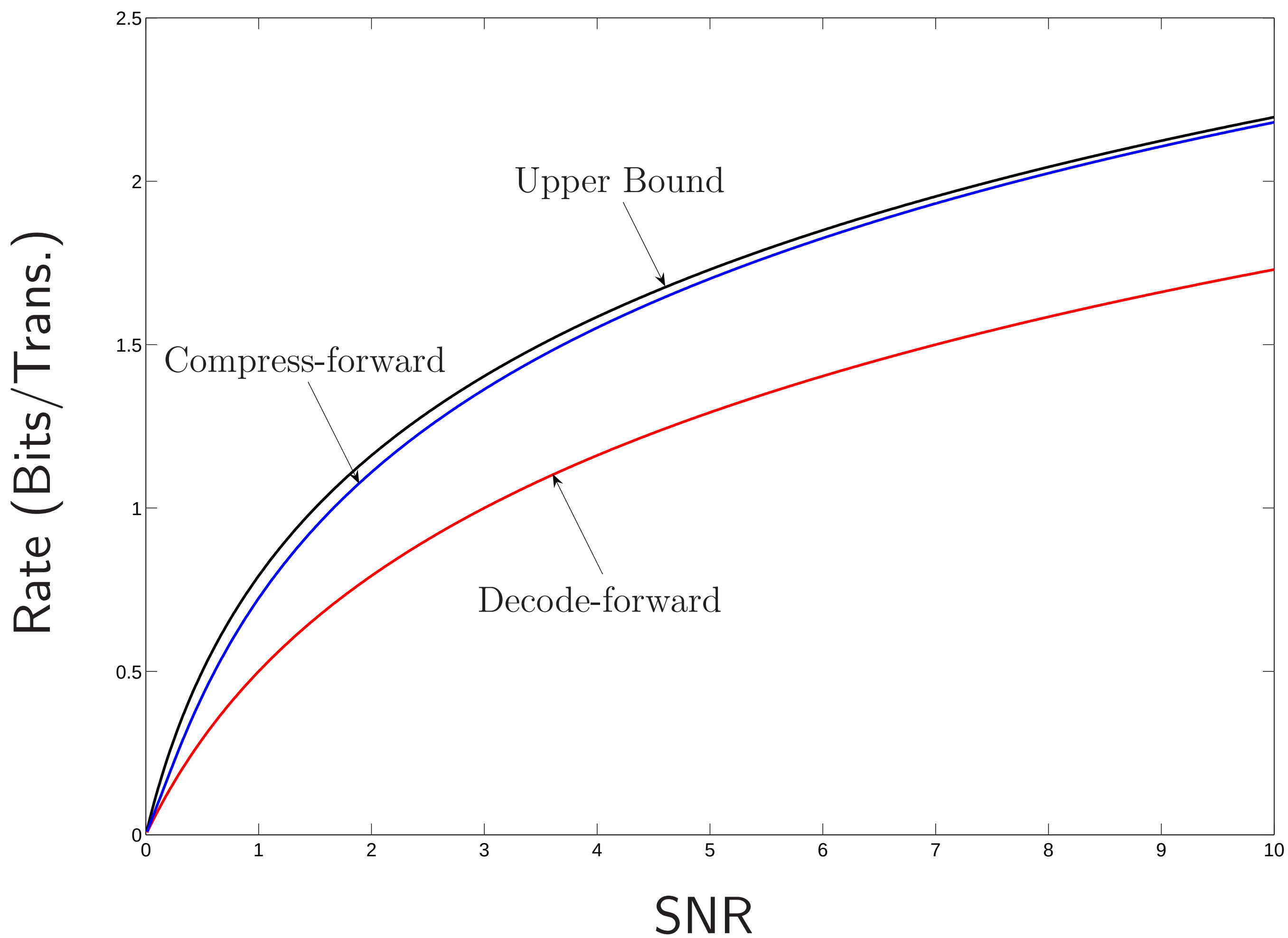


- Rate can be improved by time-sharing at the broadcast side (X sends at power P/α for fraction $0 < \alpha \leq 1$ and 0, otherwise) to:

$$C \geq \max_{0<\alpha\leq 1} \alpha \mathcal{C}\left(\frac{P}{\alpha}\left(1+\frac{a^2}{1+\left(1+\frac{a^2 P}{P+\alpha}\right)\Big/\left(-1+(1+b^2P)^{\frac{1}{\alpha}}\right)}\right)\right)$$

Comparison of Bounds for FD AWGN

- Example: $a = 1$ and $b = 2$

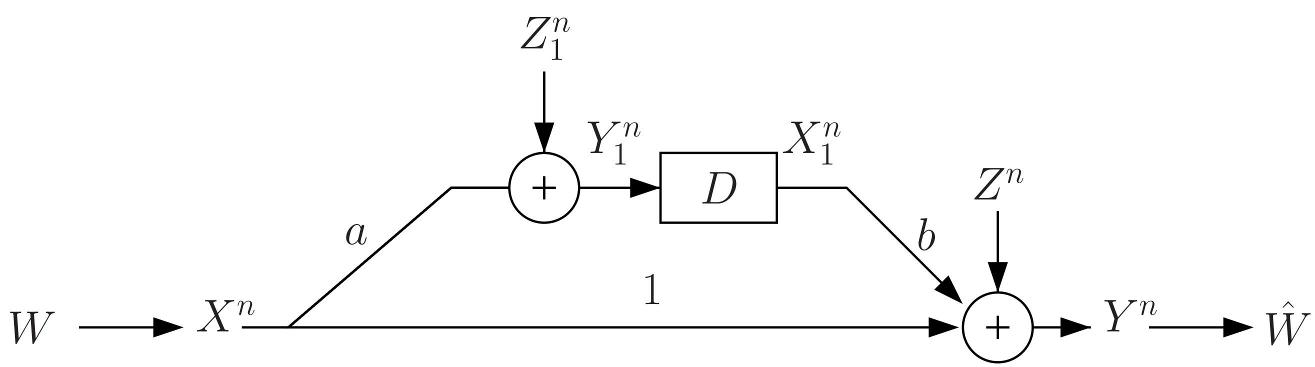


Linear Relaying

- Assume relay can only transmit a linear combination of the past received symbols, i.e., $x_{1(i+1)} = \sum_{j=1}^{i} d_{ij} y_{1j}$
- Using vector notation

$$\mathbf{X}_1 = D\mathbf{Y}_1,$$

where $D = [d_{ij}]_{k \times k}$ is a strictly lower triangular matrix and $\mathbf{X}_1 = (X_{11}, X_{12}, \ldots, X_{1k})^T$ and $\mathbf{Y}_1 = (Y_{11}, Y_{12}, \ldots, Y_{1k})^T$



- Let $C^{(l)}$ be the capacity with linear relaying subject to the power constraints

Capacity with Linear Relaying

- Capacity with linear relaying can be expressed as

$$C^{(l)} = \sup_k \frac{1}{k} C_k^{(l)} = \lim_{k \to \infty} \frac{1}{k} C_k^{(l)}$$

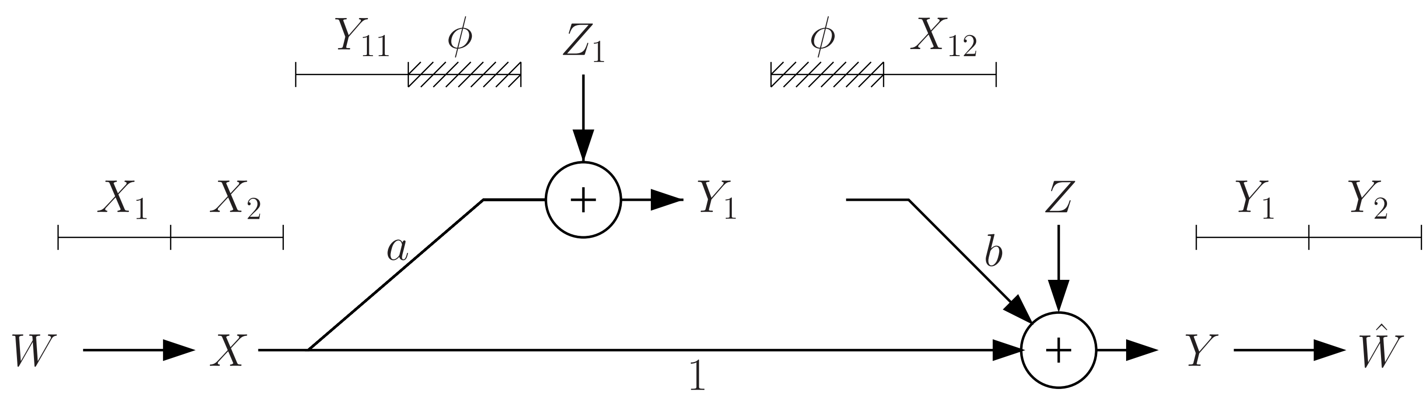
  where

$$C_k^{(l)} = \sup_{\mathrm{P}_{X^k}, D} I(X^k; Y^k)$$

  subject to

  - Sender power constraint: $\sum_{i=1}^{k} E(X_i^2) \leq kP$
  - Relay power constraint: $\sum_{i=1}^{k} E(x_{1i}^2) \leq kP$
  - Causality constraint: D strictly lower triangular

Example [EMZ'04]

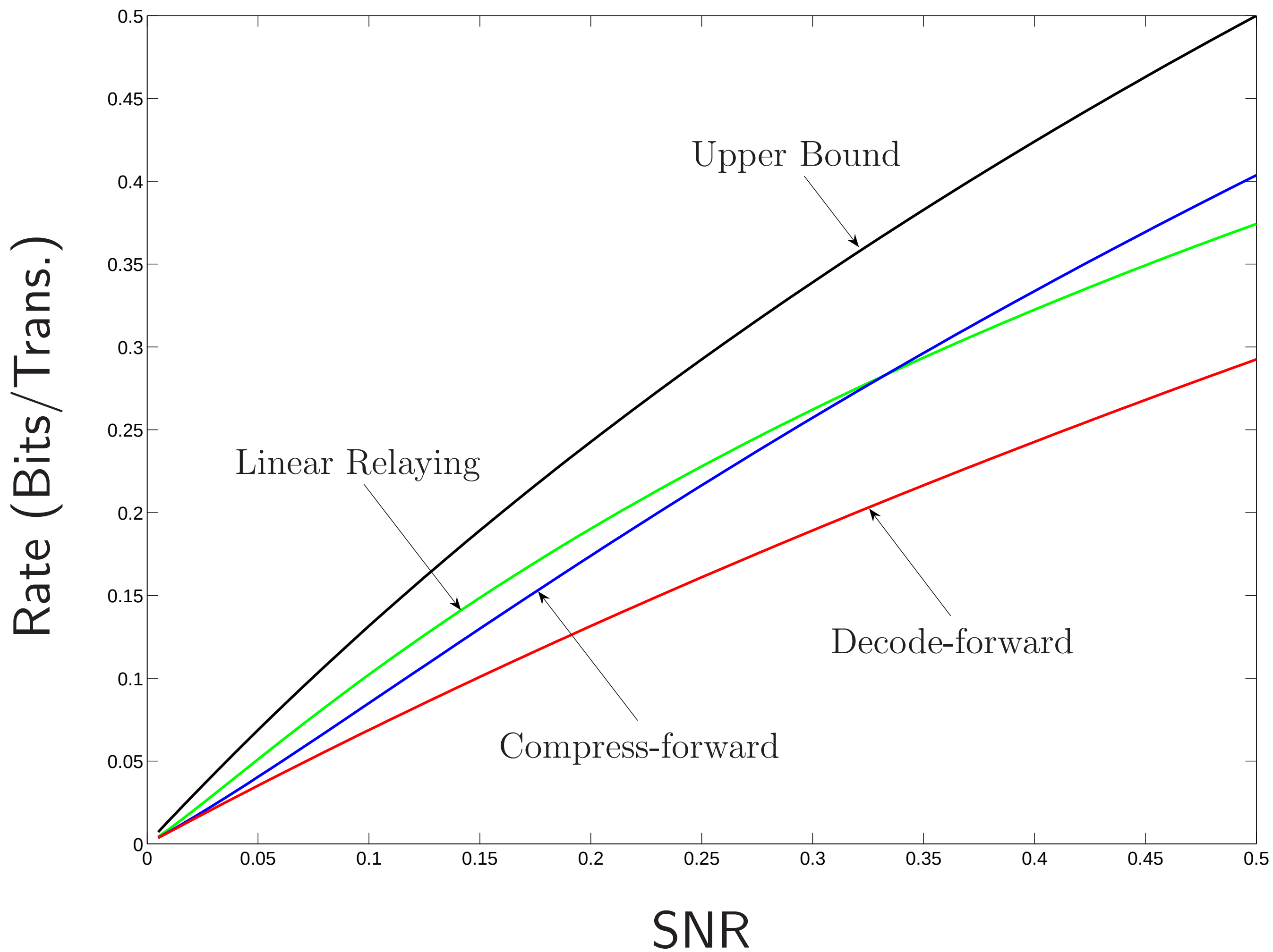


- Let $X_1 \sim \mathcal{N}(0, 2\alpha P)$, and $X_2 = \sqrt{\frac{1-\alpha}{\alpha}} X_1$ and $X_{12} = dY_{11}$ where d is chosen to satisfy relay power constraints
- The achievable rate is

$$\begin{aligned} R &= \frac{1}{2} I(X_1, X_2; Y_1, Y_2) \\ &= \max_{0 \le \alpha \le 1} \frac{1}{2} \mathcal{C} \left(2\alpha P \left(1 + \frac{\left(\sqrt{(1-\alpha)/\alpha} + abd \right)^2}{1 + b^2 d^2} \right) \right) \le \frac{1}{2} C_2^{(l)} \le C^{(l)} \end{aligned}$$

Comparison of Bounds

- Example: $a = 1$ and $b = 2$



Capacity with Linear Relaying for General AWGN

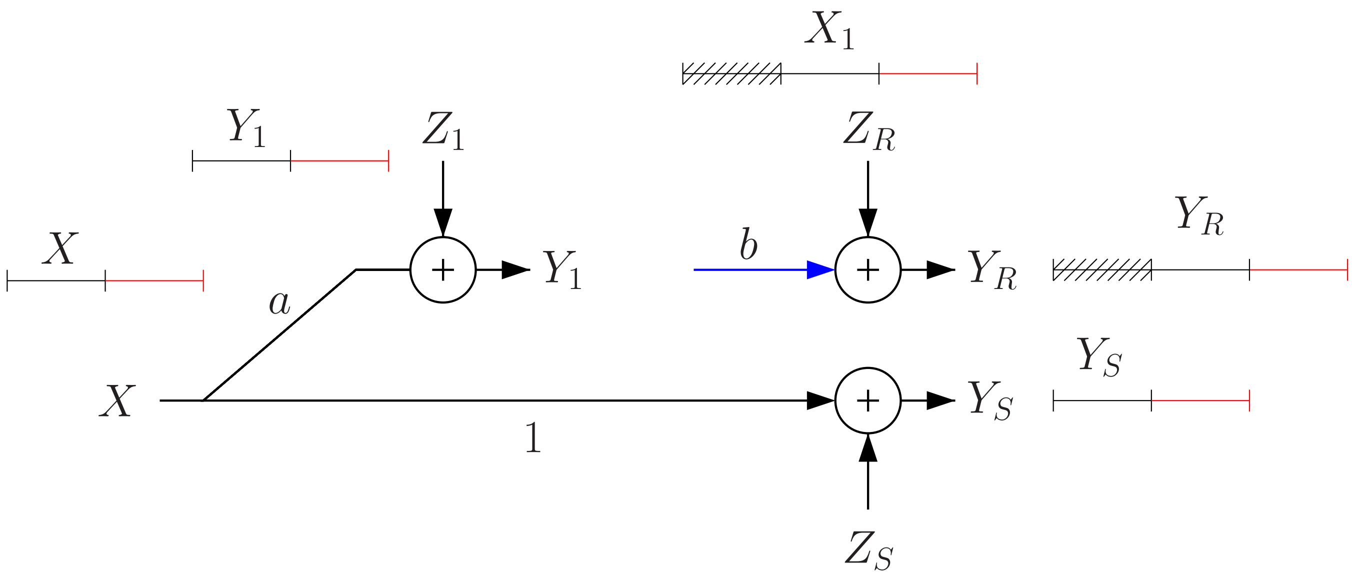
- Gaussian distribution $\mathrm{P}_{\mathbf{X}^k}$ maximizes $C_k^{(l)} = \sup_{\mathrm{P}_{X^k},D} I(X^k;Y^k)$
- Problem reduces to:

$$\begin{array}{ll} \text{Maximize} & \lim_{k\to\infty} \frac{1}{2k} \log \frac{|(I+abD)\Sigma_x(I+abD)^T + (I+b^2DD^T)|}{|(I+b^2DD^T)|} \\ \text{Subject to} & \Sigma_x \succeq 0 \\ & \text{tr}(\Sigma_x) \le kP \\ & \text{tr}(a^2\Sigma_x D^T D + D^T D) \le kP \\ & d_{ij} = 0 \text{ for } j \ge i \end{array}$$

- $\Sigma_x = E(\mathbf{X}\mathbf{X}^T)$ and D are variables of the problem
- Sequence of non-convex problems (open problem)
- "Single-letter" characterization can be found for FD-AWGN relay channel

FD-AWGN Relay Channel with Linear Relaying

- Example: Consider the following "amplify-and-forward" scheme



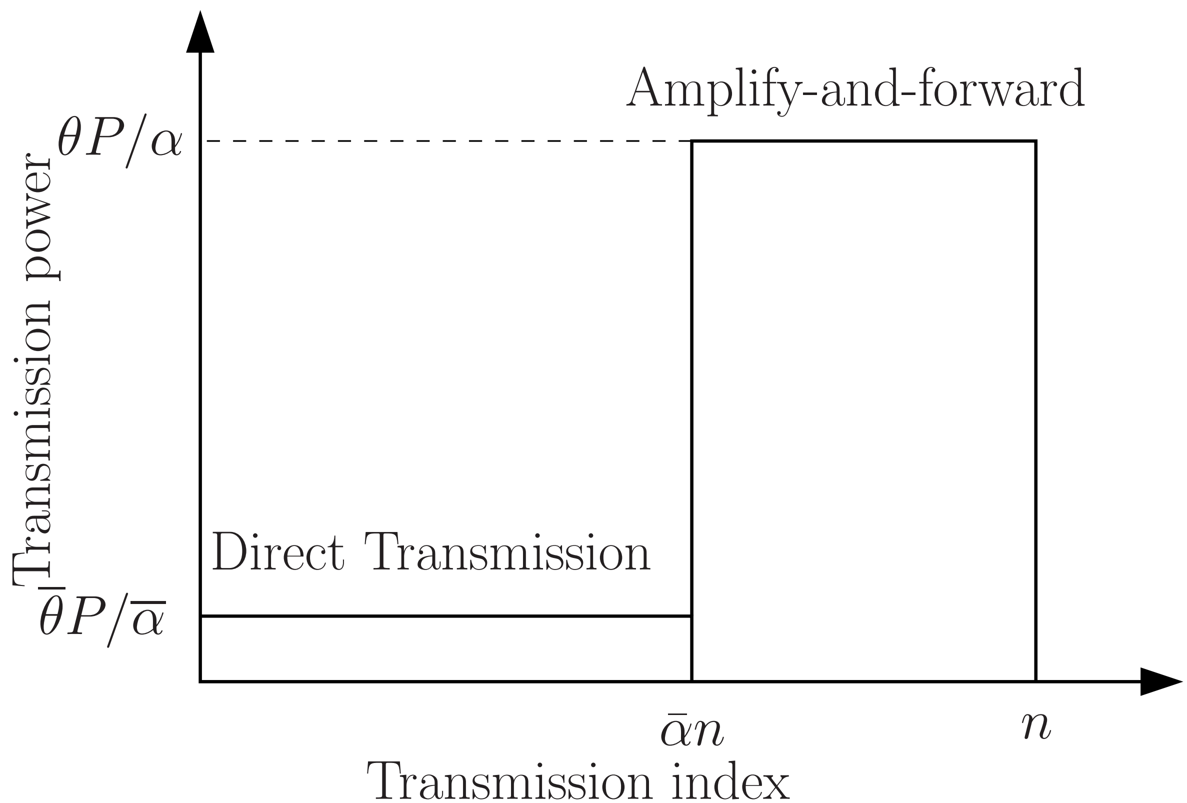
- $X \sim \mathcal{N}(0, P)$ and $X_1 = dY_1$ where d is chosen to satisfy the relay power constraint
- The achievable rate for this scheme is given by $I(X; Y_S, Y_R)$

$$C_1^{(l)} = \mathcal{C}\left(P\left(1 + \frac{a^2b^2P}{1 + (a^2 + b^2)P}\right)\right)$$

Example (continued)

- $C_1^{(l)}$ is convex for small P, concave for large P
- Rate can be improved with time-sharing on broadcast side

$$C^{\mathrm{FD-L}}(P) \geq \max_{0<\alpha,\theta\leq 1} \overline{\alpha}\mathcal{C}\left(\frac{\overline{\theta}P}{\overline{\alpha}}\right) + \alpha\mathcal{C}\left(\frac{\theta P}{\alpha}\left(1+\frac{a^2b^2P}{a^2\theta P + b^2P+\alpha}\right)\right)$$



Capacity of FD-AWGN Relay Channel with Linear Relaying

- Capacity with linear relaying for the FD-AWGN model can be expressed as

$$C^{(l)} = \sup_k \frac{1}{k} C_k^{(l)} = \lim_{k \to \infty} \frac{1}{k} C_k^{(l)},$$

where

$$C_k^{(l)} = \sup_{\mathrm{P}_{X^k}, D} I(X^k; Y_S^k, Y_R^k)$$

Subject to:

  - Sender power constraint: $\sum_{i=1}^{k} E(X_i^2) \leq kP$
  - Relay power constraint: $\sum_{i=1}^{k} E(X_{1i}^2) \leq kP$
  - Causality constraint: D lower triangular

Optimization Problem

- Gaussian input distribution $\mathrm{P}_{\mathbf{X}^k}$ maximizes $C_k^{(l)} = \sup_{\mathrm{P}_{X^k}, D} I(X^k; Y_S^k, Y_R^k)$
- Finding $C_k^{(l)}$ reduces to

Maximize $\quad \frac{1}{2k} \log_2 \frac{\left| \begin{bmatrix} I + \Sigma_x & ab\Sigma_x D^T \\ abD\Sigma_x & a^2b^2 D\Sigma_x D^T + (I + b^2 DD^T) \end{bmatrix} \right|}{\left| \begin{bmatrix} I & 0 \\ 0 & I + b^2 DD^T \end{bmatrix} \right|}$

Subject to $\quad \Sigma_x \succeq 0$

$\mathrm{tr}(\Sigma_x) \leq kP$

$\mathrm{tr}(a^2 \Sigma_x D^T D + D^T D) \leq kP$, and $d_{ij} = 0$ for $j > i$

- $\Sigma_x = E(\mathbf{X}\mathbf{X}^T)$ and lower triangular D are the optimization variables
- This is a non-convex problem in (Σ_x, D) with $k^2 + k$ variables
- For a fixed D, problem is convex in Σ_X (waterfilling optimal); however, finding D for a fixed Σ_X is a non-convex problem

Key Steps of the Proof

- Can show that diagonal Σ_x and D suffice
- Objective function reduces to

$$\text{Maximize} \quad \frac{1}{2k}\log_2 \prod_{i=1}^{k}\left(1+\sigma_i\left(1+\frac{a^2b^2d_i^2}{1+b^2d_i^2}\right)\right)$$

$$\text{Subject to} \quad \sigma_i \geq 0, \text{ for } i=1,2,\ldots,k, \quad \sum_{i=1}^{k}\sigma_i \leq kP, \text{ and}$$

$$\sum_{i=1}^{k} d_i^2(1+a^2\sigma_i) \leq kP$$

- This still is a non-convex optimization problem — time-share between k amplify-forward schemes
- At the optimum point:
  - If $\sigma_i = 0$ it is easy to show that $d_i = 0$
  - If $d_i = d_j = 0$ (direct-transmission) then $\sigma_i = \sigma_j$ (concavity of $\log$ function)

Proof (continued)

- Problem reduces to:

  Maximize $\quad \dfrac{1}{2k} \log_2 \left(1 + k\theta_0 P/k_0\right)^{k_0} \prod_{i=k_0+1}^{k} \left(1 + \sigma_i \left(1 + a^2 b^2 d_i^2/(1 + b^2 d_i^2)\right)\right)$

  Subject to $\quad d_i > 0,$ for $i = k_0 + 1, \ldots, k,$ $\displaystyle\sum_{i=k_0}^{k} \sigma_i \le k(1 - \theta_0)P,$ and

  $$\sum_{i=k_0+1}^{k} d_i^2 (1 + a^2 \sigma_i) \le kP$$

- By KKT conditions, at the optimum, there are ≤ 4 non-zero (σ_j, d_j)

  Maximize $\quad \dfrac{1}{2k} \log_2 \left(1 + k\theta_0 P/k_0\right)^{k_0} \prod_{j=1}^{4} \left(1 + \sigma_j \left(1 + a^2 b^2 d_j^2/(1 + b^2 d_j^2)\right)\right)^{k_j}$

  Subject to $\quad d_j > 0,$ for $j = 1, \ldots, 4,$ $\displaystyle\sum_{j=1}^{4} k_j \sigma_j \le k(1 - \theta_0)P,$ and

  $$\sum_{j=1}^{4} k_j d_j^2 (1 + a^2 \sigma_j) \le kP$$

Capacity of FD-AWGN Relay with Linear Relaying [EMZ'06]

- Taking the limit as $k \to \infty$, we obtain

$$C^{(l)} = \max \alpha_0 \mathcal{C}\left(\frac{\theta_0 P}{\alpha_0}\right) + \sum_{j=1}^{4} \alpha_j \mathcal{C}\left(\frac{\theta_j P}{\alpha_j}\left(1 + \frac{a^2 b^2 \eta_j}{1 + b^2 \eta_j}\right)\right),$$

  subject to $\alpha_j, \theta_j \geq 0$, $\eta_j > 0$, $\sum_{j=0}^{4} \alpha_j = \sum_{j=0}^{4} \theta_j = 1$, and $\sum_{j=1}^{4} \eta_j \left(a^2 \theta_j P + \alpha_j\right) = P$

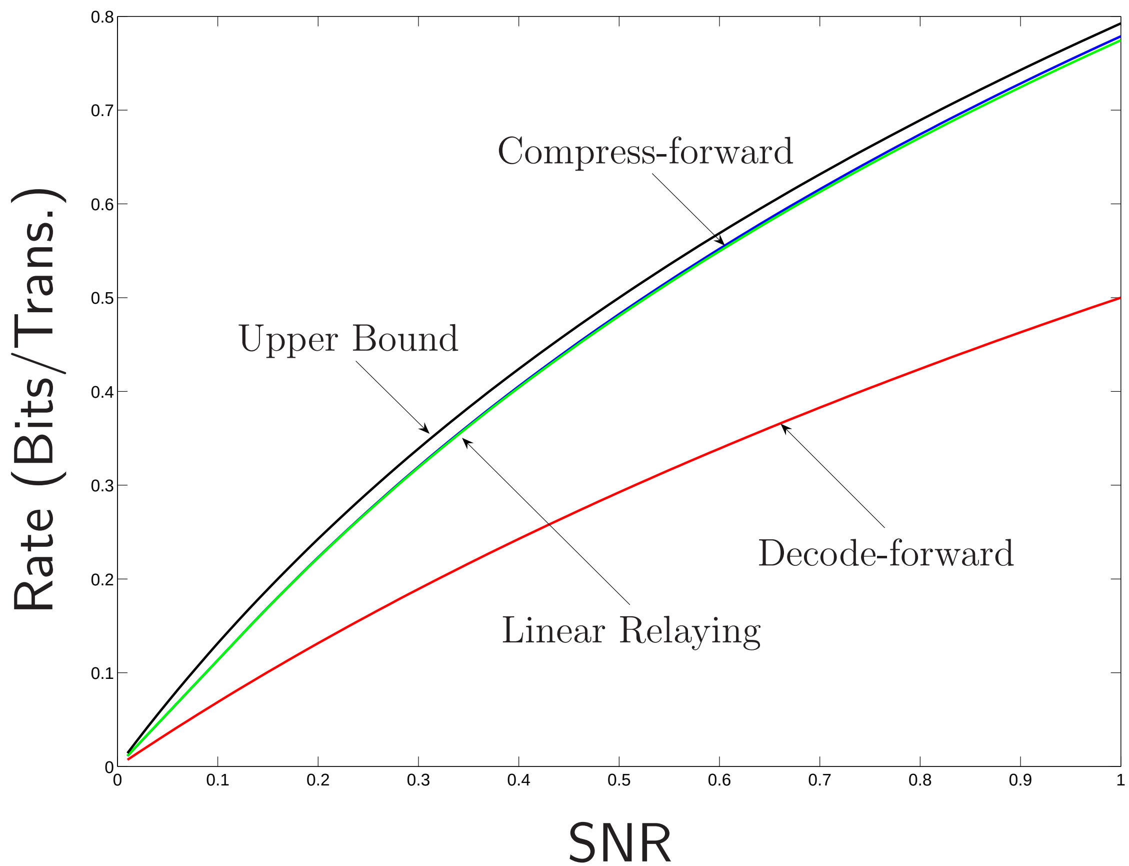
- Time-share between direct transmission and 4 amplify-forward regimes

- A non-convex optimization problem, but with only 14 variables and 3 constraints

- We haven't been able to find any example where optimal requires > 2 amplify-forward regimes (in addition to direct transmission)

Comparison with Other Schemes

- Example: $a = 1$ and $b = 5$



- As $b \to \infty$, $C^{(l)}$ approaches the cutset upper bound

Summary

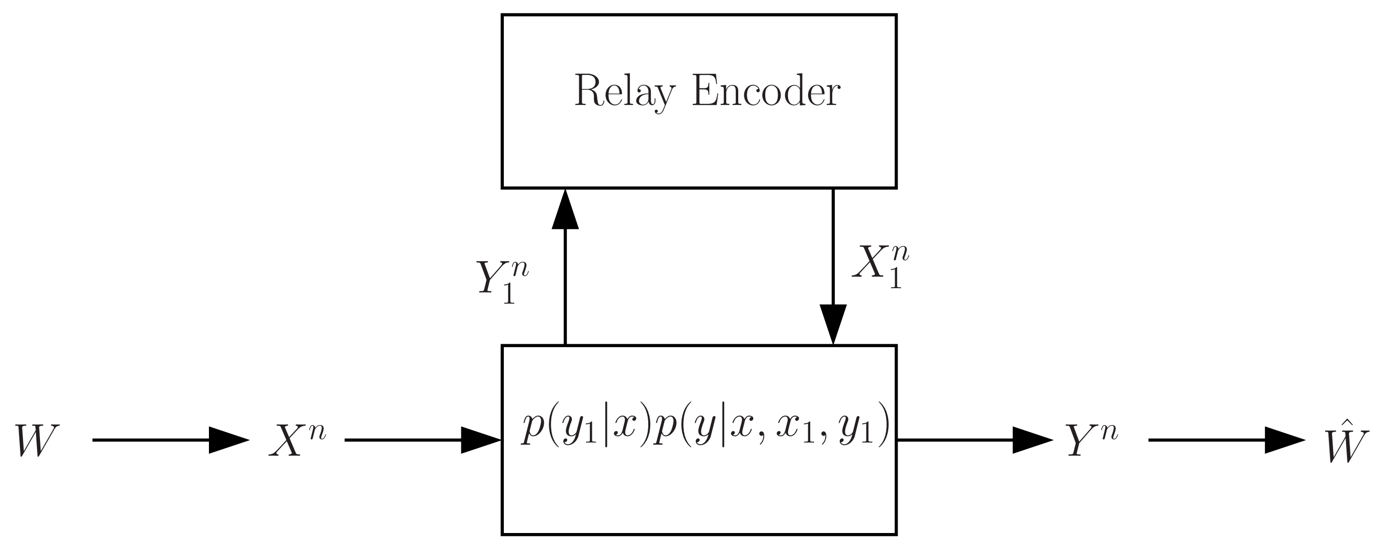
- For general AWGN relay channel:
  - Bounds are never tight for any $0 < a, b < \infty$
  - Compress-and-forward becomes optimal as $b \to \infty$
  - Linear relaying can beat compress-and-forward — we don't have a computable expression for capacity with linear relaying
- For FD-AWGN relay channel:
  - Decode-and-forward is optimal (bound coincides with cutset) when $a^2 \geq 1 + b^2 + b^2 P$
  - Compress-and-forward becomes optimal as b or $P \to \infty$. Time-sharing improves rate for small P
  - We have a computable form of capacity with linear relaying — time-sharing between direct transmission and at most 4 amplify-and-forward regimes

Relay-Without-Delay (RWD)

- Suppose the delay from the sender X to the receiver Y is longer than to the relay (X_1, Y_1), so that x_{1i} can depend on its current, in addition to past received symbols, i.e.,

$$x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1(i-1)}, y_{1i})$$

- Relay-without-delay:



- Again, wish to reliably communicate $W \in [1, 2^{nR}]$ from X to Y

Generalized Cutset Bound

- Upper bound on capacity of RWD channel:

$$C \leq \max_{p(v,x),\, f(v,y_1)} \min\{I(V,X;Y),\ I(X;Y,Y_1|V)\},$$

where $x_1 = f(v, y_1)$, and $|\mathcal{V}| \leq \min\{|\mathcal{Y}|, |\mathcal{X}| \cdot |\mathcal{X}_1|\} + 1$

- Proof follows similar lines to cutset bound:
  - Define $V_i = Y_1^{i-1}$, so $X_{1i} = f_i(V_i, Y_{1i})$
  - Key observation: $(W, Y^{i-1}) \to (X_i, V_i) \to (Y_i, Y_{1i})$
- This bound can be strictly larger than the cutset bound
- Bound applies to classical case, but
  - x_1 is a function only of v and the bound reduces to the classical cutset bound

Proof

- Use standard Fano's inequality argument.

$$nR \leq I(W;Y^n) + n\epsilon_n, \text{ where } \epsilon_n \to 0$$

- Consider

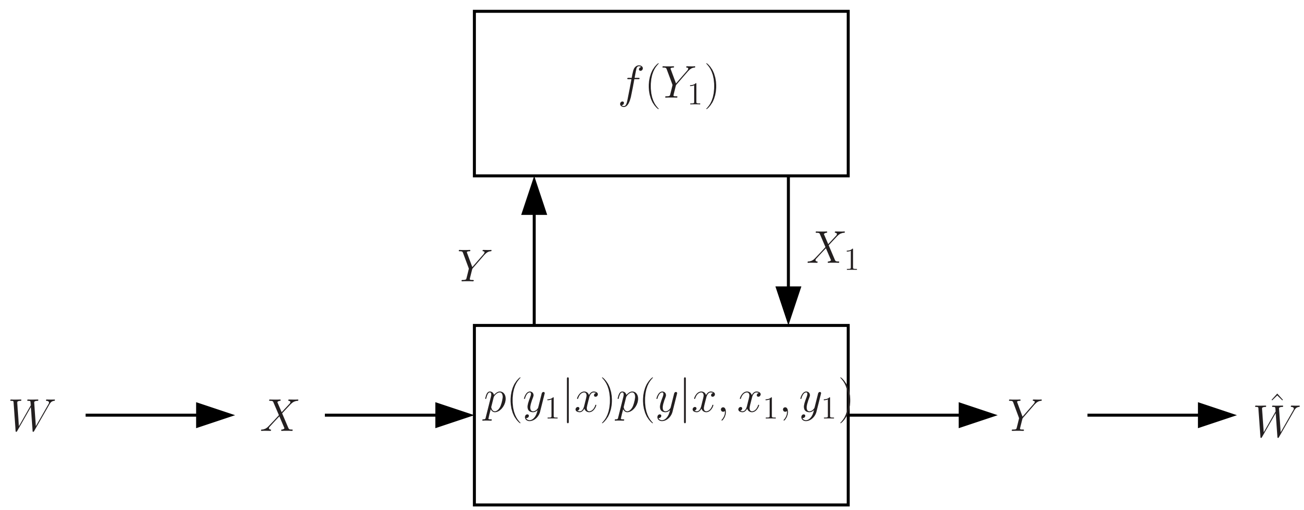
$$\begin{aligned}
I(W;Y^n) &\leq \sum_{i=1}^{n} I(W;Y_i|Y^{i-1}) \\
&\leq \sum_{i=1}^{n} (H(Y_i) - H(Y_i|W,Y^{i-1})) \\
&\leq \sum_{i=1}^{n} (H(Y_i) - H(Y_i|W,Y^{i-1},Y_1^{i-1},X_i)) \\
&= \sum_{i=1}^{n} (H(Y_i) - H(Y_i|Y_1^{i-1},X_i)) \\
&= \sum_{i=1}^{n} I(X_i,V_i;Y_i) = nI(X_Q,V_Q;Y_Q|Q) \leq nI(X,V;Y)
\end{aligned}$$

- Next consider

$$
\begin{aligned}
I(W;Y^n) &\le I(W;Y^n,Y_1^n)\\
&= \sum_{i=1}^{n} I(W;Y_i,Y_{1i}|Y^{i-1},Y_1^{i-1})\\
&= \sum_{i=1}^{n} H(Y_i,Y_{1i}|Y^{i-1},Y_1^{i-1}) - H(Y_i,Y_{1i}|Y^{i-1},Y_1^{i-1},W)\\
&\le \sum_{i=1}^{n} H(Y_i,Y_{1i}|V_i) - H(Y_i,Y_{1i}|Y^{i-1},Y_1^{i-1},W,X_i)\\
&\le \sum_{i=1}^{n} H(Y_i,Y_{1i}|V_i) - H(Y_i,Y_{1i}|V_i,X_i)\\
&= \sum_{i=1}^{n} I(X_i;Y_i,Y_{1i}|V_i) = nI(X;Y,Y_1|V)
\end{aligned}
$$

Instantaneous Relaying

- Any lower bound on the capacity of the classical relay channel, e.g., decode-and-forward, is a lower bound on the RWD channel
- Instantaneous relaying: Ignore the past and set $X_1 = f(Y_1)$;



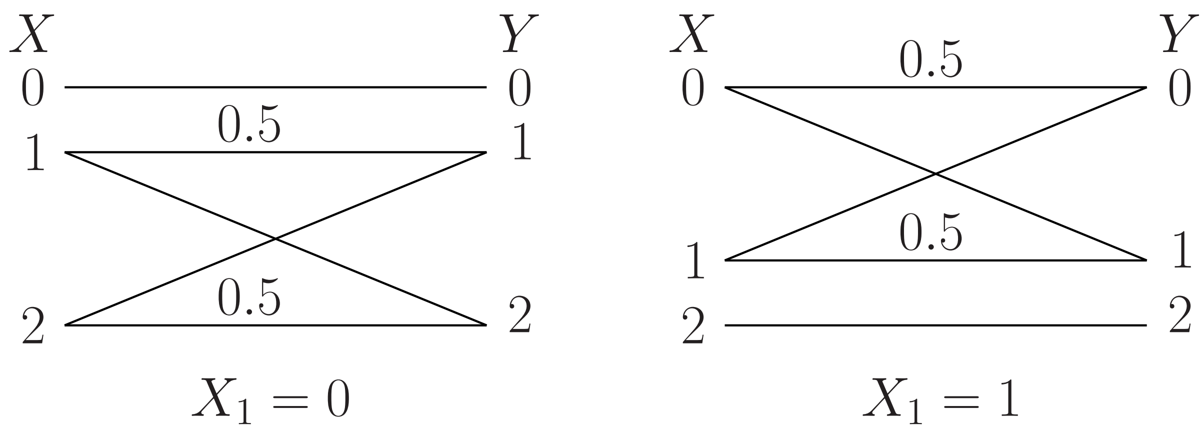
- This scheme can achieve

$$R = \max_{p(x),\, f(y_1)} I(X;Y)$$

- This strategy can be optimal and can achieve higher rate than classical cutset bound!

Sato Example

- Consider the following DW relay channel [Sato'76]. $Y_1 = X$ and



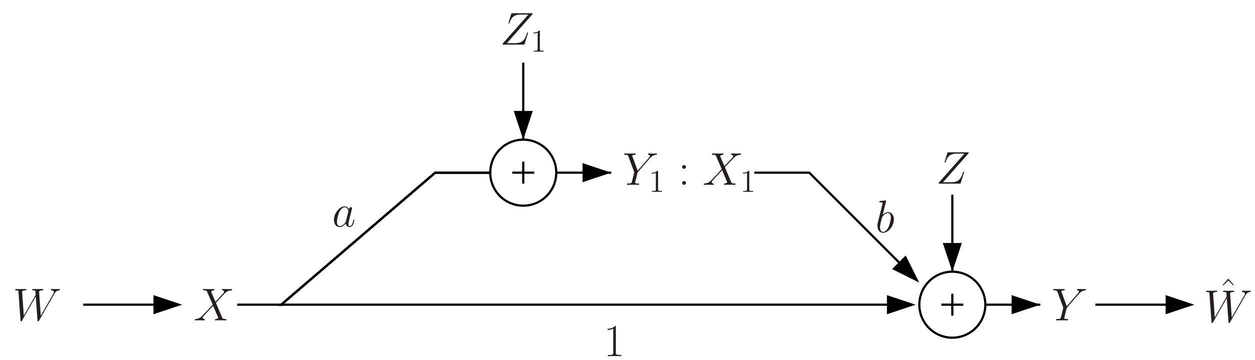
- The capacity of the classical case is 1.161878 bits/transmission [CE'79]
  - Coincides with cutset bound
- The upper bound on the capacity without delay gives:

$$C \leq 2(\log 3 - 1) = 1.169925 \text{ bits/transmission}$$

- Instantaneous relaying with input distribution $\left(\frac{3}{9}, \frac{2}{9}, \frac{4}{9}\right)$ and the mapping from X to X_1 of $0 \to 0, 1 \to 1, 2 \to 1$ achieves this bound!!
- Capacity of this RWD channel $>$ classical, which is $=$ cutset bound

AWGN RWD Channel

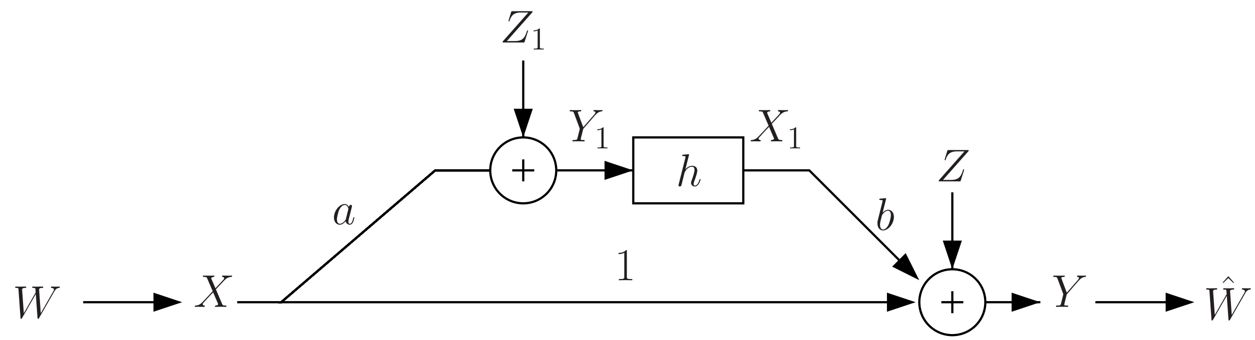
- Same model as before (but without relay coding delay)



- $a, b > 0$ are relative channel gains
- $Z_1 \sim \mathcal{N}(0, 1)$ is independent of $Z \sim \mathcal{N}(0, 1)$
- $x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1i})$
- Average power constraints: $E(X^2) \leq P$ and $E(X_1^2) \leq P$
- Recall that capacity of the classical case is not known for any $0 < a, b < \infty$

Amplify-and-Forward

- Consider the following special case of instantaneous relaying:



- If $a \leq b$, the upper bound gives

$$C \leq \mathcal{C}\left((1+a^2)P\right)$$

- Consider the case where $a^2 \leq b^2 \min\{1, P/(a^2P+1)\}$

  Amplify-and-forward also gives

$$C \geq \max_{p(x),\, E(X^2)\leq P} I(X;Y) = \mathcal{C}\left((1+a^2)P\right),$$

- Capacity in this case coincides with classical cutset bound
- Can capacity of AWGN RWD channel be $>$ cutset bound?

New Lower Bound on Capacity of RWD Channel

- Idea: Use a superposition of partial decode-and-forward and instantaneous relaying
- Achieves the lower bound:

$$C \geq \max_{p(u,v,x),\, f(v,y_1)} \min\{I(V,X;Y),\ I(U;Y_1|V) + I(X;Y|V,U)\}$$

  - U represents the information decoded by the relay in partial decode-and-forward and V (replacing X_1) represents the information sent coherently by the sender and the relay to help the receiver decode the previous U
  - Each x_1^n codeword in partial decode-and-forward is replaced by a v^n codeword, and at time i the relay sends $x_{1i} = f(v_i, y_{1i})$
- This bound coincides with the generalized cutset bound for degraded and semi-deterministic RWD channels
- Same superposition idea can be used to obain new compress-and-forward lower bound

Achievable Rate for AWGN RWD Channel

- Restrict previous scheme to superposition of decode-and-forward and amplify-and-forward:
  - Let $U = X = V + X'$, where $V \sim \mathcal{N}(0, \alpha P)$ and $X' \sim \mathcal{N}(0, \overline{\alpha} P)$ are independent, for $0 \le \alpha \le 1$ and $\overline{\alpha} = 1 - \alpha$
  - Let X_1 be a normalized convex combination of Y_1 and V

$$X_1 = h(\beta Y_1 + \overline{\beta} V), \; 0 \le \beta \le 1$$

- Using the power constraint on the relay sender, we obtain

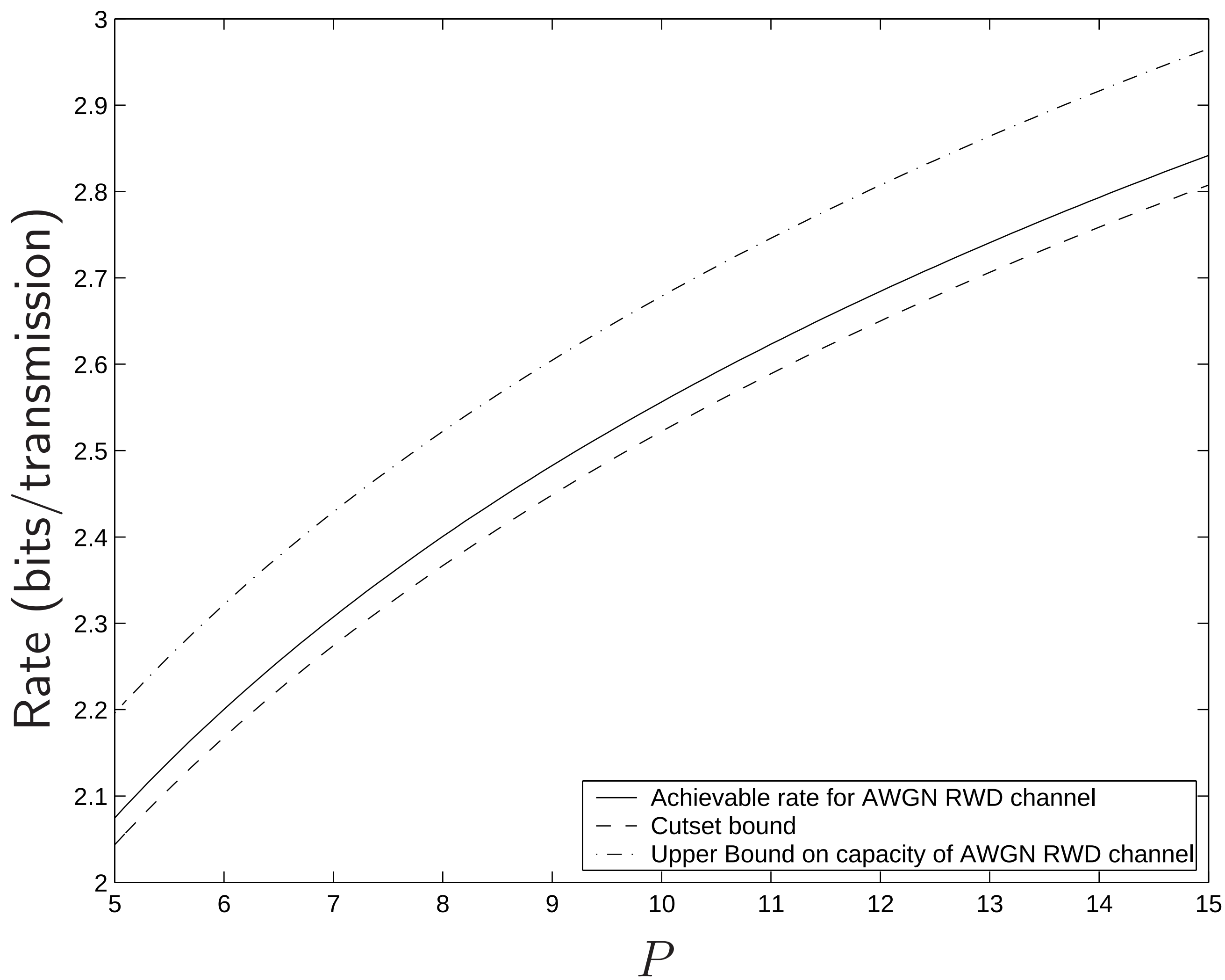
$$h^2 = \frac{P}{(\overline{\beta} + a\beta)^2 \alpha P + \beta^2 (a^2 \overline{\alpha} P + N)}$$

- Substituting the above choice of (U, V, X, X_1), we obtain

$$C_{\text{RWD}-\text{AWGN}} \ge \max_{\alpha, \beta} \min \left\{ \mathcal{C}\left(a^2 \overline{\alpha} P\right), \mathcal{C}\left(\frac{\alpha P (bh(a\beta + \overline{\beta}) + 1)^2 + \overline{\alpha} P (bha\beta + 1)^2}{(1 + (\beta b h)^2)} \right) \right\}$$

Achievable Rate vs. Cutset vs. Upper Bound

Assume $a = 2$ and $b = 1$



Classical vs. RWD Relay Channels

Result	Classical Relay	RWD
Capacity	Not known in general	Not known in general Can be $>$ classical
Upper bound	$\max_{p(x,x_1)} \min\{I(X,X_1;Y),\ I(X;Y,Y_1\|X_1)\}$ Cutset bound	$\max_{p(v,x),\ f(v,y_1)} \min\{I(V,X;Y),\ I(X;Y,Y_1\|V)\}$ Can be $>$ cutset bound
Degraded capacity $X \to (X_1,Y_1) \to Y$	$\max_{p(x,x_1)} \min\{I(X,X_1;Y),\ I(X;Y_1\|X_1)\}$ Decode-forward	$\max_{p(v,x),\ f(v,y_1)} \min\{I(V,X;Y),\ I(X;Y_1\|V)\}$ Decode-forward $+$ instantaneous coding
Semi-det capacity $Y_1 = g(X)$	$\max_{p(x,x_1)} \min\{I(X,X_1;Y),\ I(X;Y,Y_1\|X_1)\}$	$\max_{p(v,x),\ f(v,y_1)} \min\{I(V,X;Y),\ I(X;Y,Y_1\|V)\}$ Partial decode-forward $+$ instantaneous coding
AWGN	Capacity not known for any $a,b \neq 0$	Capacity known for $a^2 \leq b^2 \min\{1, \frac{P}{a^2P+N}\}$ Amplify-forward Can be in general $>$ classical

Conclusion

- Overview of known upper and lower bounds on relay and RWD channels
  - Bounds are tight only is special cases
  - We know as much about the RWD as classical relay, main difference is instantaneous relaying, but it is not sufficient
- Generalized cutset bound needed for RWD channel
  - Generalization to relay networks with delays (see poster with J. Mammen)