Relay Networks with Delays

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Delay

- Delay (and its variation) are main concerns in electronic systems
- Delay (and its variation) are main concerns in communication networks
  - Delay in forward link hurts interactivity
  - Delay in feedback loop leads to instability
  - Delay variation leads to the need for large buffers
- Delay is also a main concern in control systems:
  - Delay in feedback loop can result in instability
  - Early noisy measurement is better than a later more accurate one
Delay

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• Delay (and its variation) are main concerns in communication networks
  ◦ Delay in forward link hurts interactivity
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• Delay is also a main concern in control systems:
  ◦ Delay in feedback loop can result in instability
  ◦ Early noisy measurement is better than a later more accurate one

• What is the effect of delay on information capacity of networks?
  ◦ Since we let coding delay \( \to \infty \), does delay matter?

  We investigate this question for feedforward relay networks
DMC with Delay

- DMC with delay $d(1, 2) \in \{0, 1, \ldots\}$

\[
\begin{align*}
X_t^1 & \xrightarrow{\ p(y^2|x^1)\ } d(1, 2) \xrightarrow{\ } Y_{t+d(1,2)}^2
\end{align*}
\]

- Channel is memoryless in the sense that for any block length $n \geq 1$,

\[
p(y_{d(1,2)+1:d(1,2)+n}^2|x_{1:n}^1) = \prod_{t=d(1,2)+1}^{d(1,2)+n} p(y_t^2|x_{t-d(1,2)}^1)
\]

- $d(1, 2) = 0$ gives the classical DMC

- Denote channel capacity by $C_d$

- Clearly $C_d = C_0$ $\Rightarrow$ delay doesn’t change capacity

- Graphical representation:
BC with Delay

- DM-BC with delay

- Channel is memoryless and $Y^2, Y^3$ are independent given $X^1$:

$$p(y^2_{d(1,2)+1:d(1,2)+n}, y^3_{1:n} | x^1_{1:n}) = \prod_{t=d(1,2)+1}^{d(1,2)+n} p(y_t^2 | x^1_{t-d(1,2)}) \prod_{t=1}^{n} p(y_t^3 | x^1_t)$$

- $d(1, 2) = 0$ gives the classical BC

- Capacity region doesn’t depend on delay
MAC with Delay

- DM-MAC with delay

\[ d(1, 3) = 0 \]

- Channel is memoryless:

\[
p(y_{1:d(1,3)+n}^3|x_{1:n}^1, x_{1:n}^2) = \prod_{t=1}^{d(1,3)+n} p(y_t^3|x_t^1, x_{t-d(1,3)}^2),
\]

where \( x_t^1 \) and \( x_{t-d(1,3)}^2 \) are arbitrary if \( t \) is outside specified ranges

- \( d(1, 3) = 0 \) gives the classical MAC

- Again capacity region doesn’t depend on delay [CMP’81]
• DM relay channel

- Relay transmission at $t = 1, 2, \ldots$ can depend only on past received symbols: $X^2_t = f_t(Y^2_1, Y^2_2, \ldots, Y^2_{t-1}) = f_t(Y^2_{1:t-1})$

- Equivalently, set $X^2_t = f_t(Y^2_{2:t})$ and add link delay $d(1, 2) = 1$

- Capacity $C_1$ known only for some classes
Classical Cut-Set Bound [CE’79]

- Upper bound on capacity of classical relay:

\[ C_1 \leq \max_{p(x^1,x^2)} \min \{ I(X^1, X^2; Y^3), I(X^1; Y^3, Y^2|X^2) \} \]

- Tight for all classes where capacity is known
- Not known if it is tight in general — no known tighter bound
- Can be generalized to relay networks [E’81] and more general networks [CT’91]
Relay With and Without Delay

- Classical relay channel (capacity $C_1$)

\[
d(1, 2) = 1 \\
d(2, 3) = 0 \\
d(1, 3) = 0
\]
Relay With and Without-Delay

- Classical relay channel (capacity $C_1$)

- Relay-without-delay [EH’05] (capacity $C_0$)

- Is $C_0 = C_1$? (Clearly $C_0 \geq C_1$)

- Does the cut-set bound apply to the relay-without-delay?
Sato 1976 Example

• DM relay channel with $Y^2 = X^1$ and

  $X^1$
  0  0.5  1
  1  0.5  2
  2

  $Y^3$
  0  1  0
  0  1  2

  $X^2 = 0$

  $X^2 = 1$

• Capacity for the classical case $C_1 = 1.161878$ bits/trans [CE’79]
  ○ Coincides with classical cut-set bound
Sato 1976 Example

- DM relay channel with $Y^2 = X^1$ and

  $X^1 \begin{array}{c|c|c}
  0 & 0.5 & 0 \\
  1 & 0.5 & 1 \\
  2 & 0.5 & 2 \\
\end{array}$

  $Y^3 \begin{array}{c|c|c}
  0 & 0 \\
  1 & 1 \\
  2 & 2 \\
\end{array}$

  $X^2 = 0$  \hspace{1cm}  $X^2 = 1$

- Capacity for the classical case $C_1 = 1.161878$ bits/trans [CE’79]
  - Coincides with cut-set bound

- Capacity without delay $C_0 = 2(\log 3 - 1) = 1.169925$ bits/trans [EH’05]
  - Achieved using instantaneous relay coding (set $X^2_t = f(Y^2_t)$)
  - $C_0 >$ cut-set bound!!

- Capacity without delay can be $>$ with delay

- Cut-set bound doesn’t apply to relay-without-delay!!
New Cut-set Bound [EH’05]

- Cut-set bound for relay-without-delay channel:

\[ C_0 \leq \max_{p(x^1,u), f(u,y^2)} \min \{ I(X^1,U;Y^3), I(X^1;Y^2,Y^3|U) \}, \]

where \( x_1 = f(u,y^2) \), and \( |U| \leq \min \{|Y^3|, |X^1| \cdot |X^2|\} + 1 \)

- Proof follows similar lines to cut-set bound:

  - Define: \( U_t = Y^2_{1:t-1} \), so \( X^2_t = f_t(U_t, Y^2_t) \)
  - Key observation: \( (W,Y^3_{1:t-1}) \rightarrow (X^1_t,U_t) \rightarrow (Y^3_t,Y^2_t) \)

- The bound can be strictly larger than cut-set bound (e.g., Sato Ex.)
- The bound reduces to classical cut-set for classical relay
# Classical Relay vs. Relay-without-delay

<table>
<thead>
<tr>
<th>Result</th>
<th>Classical Relay [CE’79]</th>
<th>RWD [EH’05]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacity</td>
<td>Not known in general</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>Can be $&gt;$ classical</td>
</tr>
<tr>
<td>Upper bound</td>
<td>$\max_{p(x^1,x^2)} \min {I(X^1, X^2; Y^3), I(X^1; Y^3, Y^2</td>
<td>X^2)}$</td>
</tr>
<tr>
<td></td>
<td>Classical cut-set bound</td>
<td>New cut-set bound $\geq$ Classical</td>
</tr>
<tr>
<td>Lower bound</td>
<td>$\max_{p(v,x^1,x^2)} \min {I(X^1, X^2; Y^3), I(X^1; Y^3</td>
<td>X^2, V) + I(V; Y^2</td>
</tr>
<tr>
<td></td>
<td>Generalized Block Markov</td>
<td>Generalized Block Markov + instant coding</td>
</tr>
<tr>
<td>Semi-det capacity</td>
<td>$\max_{p(x^1,x^2)} \min {I(X^1, X^2; Y^3), I(X^1; Y^2, Y^3</td>
<td>X^2)}$</td>
</tr>
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<td>$Y^2 = g(X^1)$</td>
<td>Generalized block Markov</td>
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</tr>
</tbody>
</table>
DM Relay Network with Delays

- Weighted directed acyclic graph (DAG) \((\mathcal{N}, \mathcal{E}, d)\):

```
2 -> i -> j
  \[ d(1, 2) \]
1 -> 4 -> 3
  \[ d(1, 3) \]
  \[ d(1, 4) \]
```

- \(X^1\) associated with node 1 (sender), \(Y^K\) associated with node K (receiver), \((X^i, Y^i)\) associated with node \(i\) (relays)

- Family of conditional pmfs: \(\{p(y^i | \{x^j : j \in \mathcal{N}_i\}) , i = 2, 3, \ldots, K\}\)

  where \(\mathcal{N}_i = \{j \in \mathcal{N} : (j, i) \in \mathcal{E}\}\)

- Delay \(d(i, j) \in \{0, 1, \ldots\}\) assigned to edge \((i, j) \in \mathcal{E}\)

- Network is “memoryless”
DM Relay Network with Delays

- Let $d(i)$, $D(i)$ be shortest and longest path delay from 1 to $i$, respectively.

- $(2^{nR}, n)$ code consists of:
  1. a message $W \in \{1, 2, \ldots, 2^{nR}\}$,
  2. encoding function that maps message $W$ into codeword $X^1_{1:n}(w)$,
  3. relay encoding functions $X^i_t = f^i_t \left(Y^{i}_{d(i)+1:t}\right)$, $i = 2, \ldots, K - 1$, and
  4. decoding function that maps $Y^K_{d(K)+1:D(K)+n}$ into an estimate $\hat{W}$

- Network capacity $C$ is the supremum over the set of achievable rates
Two Results

• How does capacity depend on the link delays?
  ◦ We already know that delay matters

• We present two preliminary results:
  1. Sufficient conditions for two networks with the same graph and associated random variables to have the same capacity
  2. New cut-set bound that reduces to classical cut-set for classical relay and to the new cut-set bound (with $U$) for relay-without-delay
Sufficient Conditions for Equal Capacity

- Consider two relay networks with same DAG, same set of associated random variables, and same conditional pmfs, but different link delays.

Let $w_i(p)$ for network $i = 1, 2$ be delay of path $p$ from 1 to $K$.

Capacities equal if $\forall p$, $w_1(p) - w_2(p) = m$, for some integer $m$.

- Examples:
We show that any \((2^{nR}, n)\) code for one network can be mapped into a code for the other with the same error probability. Code mapping:

(i) Use same encoding function for node 1
(ii) Use same relay encoding functions with appropriate time shifts
(iii) Use same decoding function at node \(K\) with appropriate time shift

For example:

\[
\begin{align*}
X_{t+3}^{1,2} &= f_{t+3}^{1,2} \\
X_{1:n}^{1,1} &= 0 \\
Y_{t+3}^{1,4} &= 0 \\
f_{t+1}^{1,3} &= 1 \\
Y_{t+5}^{2,4} &= 1 \\
X_{1:n}^{1,1} &= 4 \\
f_{t+4}^{2,2} &= f_{t+1}^{1,2} \\
f_{t+4}^{2,3} &= f_{t+1}^{1,3}
\end{align*}
\]
General Cut-Set Bound

- Upper bound is of the form:
  \[ C \leq \sup_{S \subset N} \min \{ I(X(S), U(S); Y(S^c)|X(S^c), U(S^c)) \} \]

- Node 1 in \( S \) and node \( K \) is in \( S^c \)
- Supremum over all joint pmf of r.v.s in \( X(S), U(S), X(S^c), U(S^c) \) for all \( S \) and over all relay functions involving a \( U \)
- The random variables constituting \( X(S), U(S), X(S^c), U(S^c) \) are not unique—can write different \( I \) expressions for same cut
- The procedure in the paper uses the structure of the network to reduce the number of random variables used
- Bound reduces to cut-set bounds for classical and relay-without-delay
- Two new ingredients beyond classical cut-set:
  - Assigning \( U \)s to some nodes instead of \( X \)s (as in RWD bound)
  - Using multiple \( X \) r.v.s for same sender
When do We Assign a $U$?

- Recall classical and relay-without-delay bounds:

  \[
  C_1 \leq \max_{p(x^1,x^2)} \min \left\{ I(X^1,X^2;Y^3), I(X^1;Y^3,Y^2|X^2) \right\}
  \]

  \[
  C_0 \leq \max_{p(x^1,u), f(u,y^2)} \min \left\{ I(X^1,U;Y^3), I(X^1;Y^2,Y^3|U) \right\}
  \]

- Note that in RWD node 2 is on shortest path from 1 to 3; but not on shortest path in classical

- In general, we assign $U^i$ when node $i \in S$ is on shortest path from node 1 to node $j \in S^c$; otherwise we assign $X^i$
Example

- 2-relay network:

- Relay node 2 is on shortest path, so it is assigned an auxiliary random variable $U^2$; relay node 3 isn’t, so it is assigned $X^3$

- Cut-set bound is of the form: $C \leq \sup_{p(x^1,u^2,x^3),f(u^2, y^2)} \min\{I_1, I_2, I_3, I_4\}$,

  \[ I_1 = I(X^1; Y^2, Y^3|U^2, X^3), \quad I_2 = I(X^1, U^2; Y^3, Y^4|X^3), \]

  \[ I_3 = I(X^1, X^3; Y^2, Y^4|U^2), \quad I_4 = I(U^2, X^3; Y^4) \]
When Do We Use More RVs for $X^i$?

- 2-relay network: Consider cut $C$

Note that $Y^3_{t+1}$ depends on $X^1_{t-1}$ while $Y^2_t$ depends on $X^1_t$. There is an edge from 2 to 3, so $Y^3_{t+1}$ also depends indirectly on $X^1_t$.

- Need to include 2 r.v.s for $X^1$ in the inequality for this cut

- This cut gives the bound

$$C \leq I(X^1_1, X^1_2; Y^2_2, Y^3_3 | U^2_2, X^3_1)$$

Note: Replace $X^3_3$ with $X^3_1$ in paper
Conclusion

- Delay can change network information capacity
  - Relative path delay seems to be what matters
- Simplest example is classical relay versus relay-without-delay [EH’05]:
  - No delay makes it possible to perform *instantaneous* relaying
  - Instantaneous relaying can be optimal (Sato Ex.[EH’05])
  - Instantaneous relaying can beat classical cut-set bound (Sato Ex.)
- Two new results for DM relay with delay networks:
  - Sufficient conditions for two networks to have same capacity
  - General cut-set bound for relay networks with delays
    New ingredients:
    - Assign a $U$ when relay node is on shortest path from node 1
    - Assign more than one random variable for same node
- More work needed to better understand the role of delay