Relay Networks with Delays

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Delay

- Delay (and its variation) are main concerns in electronic systems
- Delay (and its variation) are main concerns in communication networks
  - Delay in forward link hurts interactivity
  - Delay in feedback loop leads to instability
  - Delay variation leads to the need for large buffers
- Delay is also a main concern in control systems:
  - Delay in feedback loop can result in instability
  - Early noisy measurement is better than a later more accurate one

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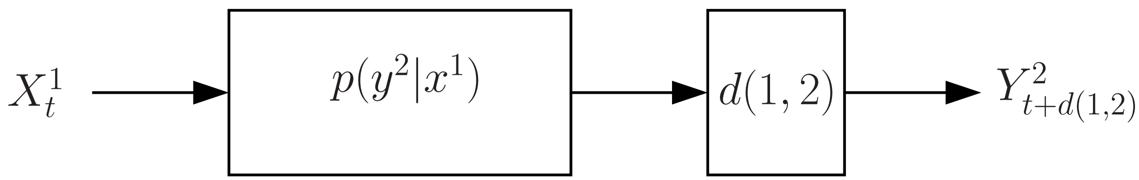
- Delay is also a main concern in control systems:
  - Delay in feedback loop can result in instability
  - Early noisy measurement is better than a later more accurate one
- What is the effect of delay on information capacity of networks?
  - Since we let coding delay $\to \infty$, does delay matter?

We investigate this question for *feedforward* relay networks

DMC with Delay

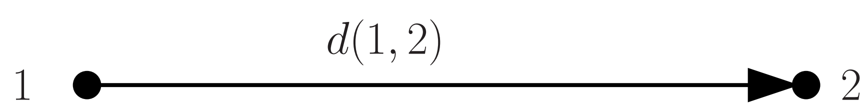
- DMC with delay $d(1,2) \in \{0, 1, \ldots\}$



- Channel is memoryless in the sense that for any block length $n \geq 1$,

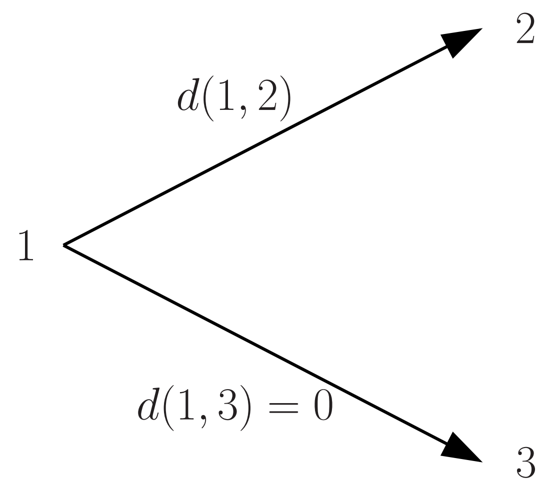
$$p(y^2_{d(1,2)+1:d(1,2)+n}|x^1_{1:n}) = \prod_{t=d(1,2)+1}^{d(1,2)+n} p(y_t^2|x^1_{t-d(1,2)})$$

- $d(1,2) = 0$ gives the classical DMC
- Denote channel capacity by C_d
- Clearly $C_d = C_0 \Rightarrow$ delay *doesn't* change capacity
- Graphical representation:



BC with Delay

- DM-BC with delay



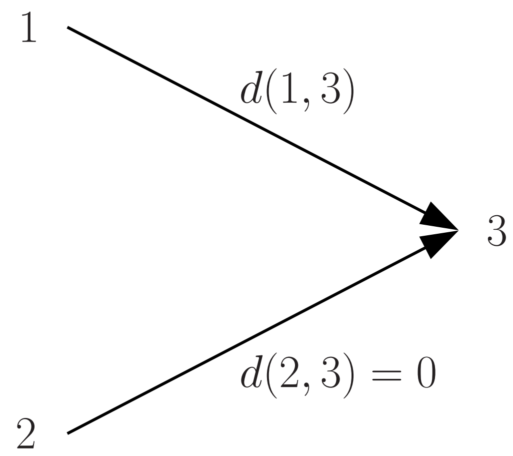
- Channel is memoryless and Y^2, Y^3 are independent given X^1:

$$p(y^2_{d(1,2)+1:d(1,2)+n},\ y^3_{1:n}|x^1_{1:n}) = \prod_{t=d(1,2)+1}^{d(1,2)+n} p(y^2_t|x^1_{t-d(1,2)}) \prod_{t=1}^{n} p(y^3_t|x^1_t)$$

- $d(1,2) = 0$ gives the classical BC
- Capacity region doesn't depend on delay

MAC with Delay

- DM-MAC with delay



- Channel is memoryless:

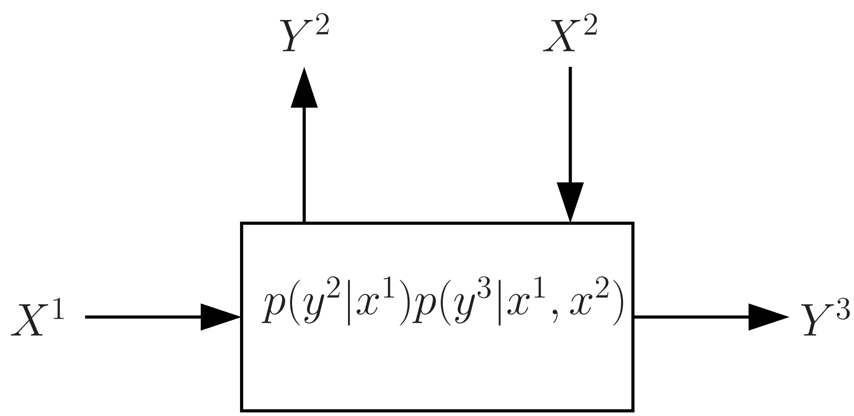
$$p(y^3_{1:d(1,3)+n} | x^1_{1:n}, x^2_{1:n}) = \prod_{t=1}^{d(1,3)+n} p(y^3_t | x^1_t, x^2_{t-d(1,3)}),$$

  where x^1_t and $x^2_{t-d(1,3)}$ are arbitrary if t is outside specified ranges

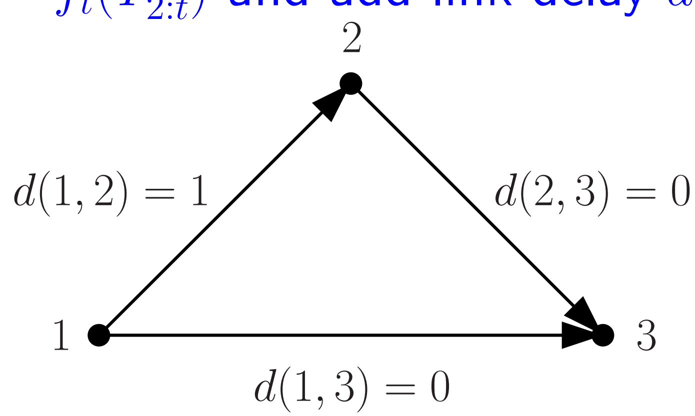
- $d(1,3) = 0$ gives the classical MAC
- Again capacity region doesn't depend on delay [CMP'81]

Classical Relay Channel

- DM relay channel



- Relay transmission at $t = 1, 2, \ldots$ can depend only on *past received symbols*: $X_t^2 = f_t(Y_1^2, Y_2^2, \ldots, Y_{t-1}^2) = f_t(Y_{1:t-1}^2)$
- Equivalently, set $X_t^2 = f_t(Y_{2:t}^2)$ and add link delay $d(1, 2) = 1$

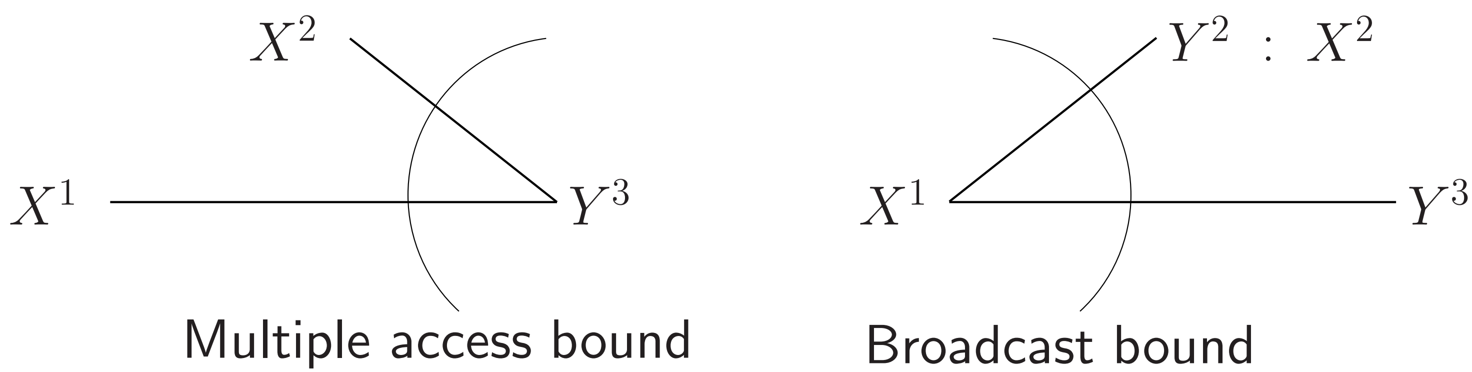


- Capacity C_1 known only for some classes

Classical Cut-Set Bound [CE'79]

- Upper bound on capacity of classical relay:

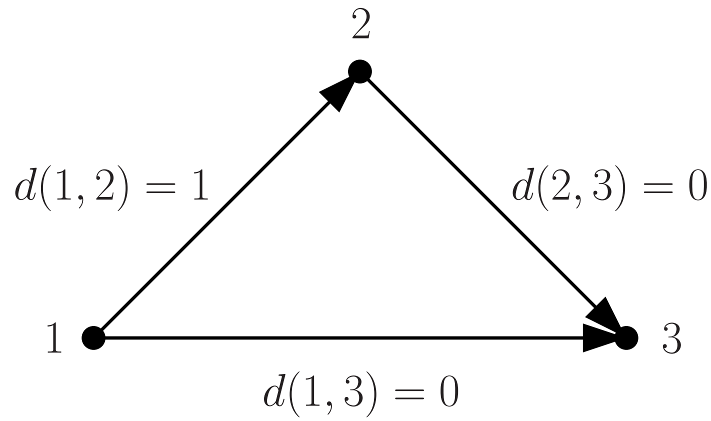
$$C_1 \leq \max_{p(x^1,x^2)} \min \left\{ I(X^1, X^2; Y^3), I(X^1; Y^3, Y^2 | X^2) \right\}$$



- Tight for all classes where capacity is known
- Not known if it is tight in general — no known tighter bound
- Can be generalized to relay networks [E'81] and more general networks [CT'91]

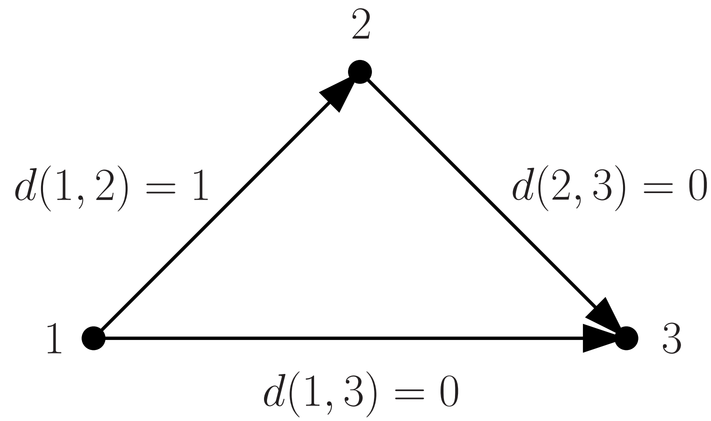
Relay With and Without Delay

- Classical relay channel (capacity C_1)

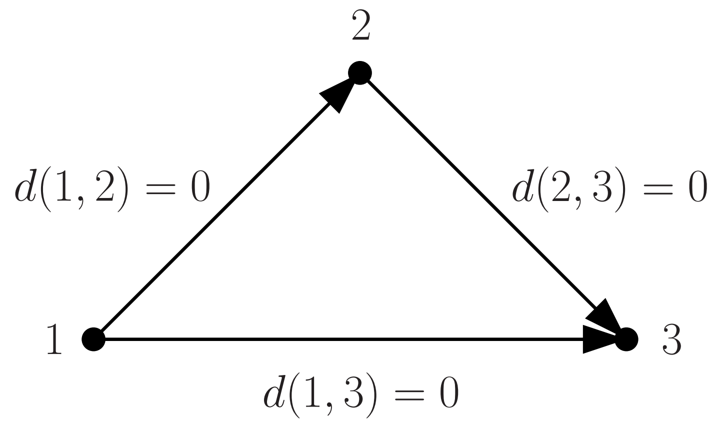


Relay With and Without-Delay

- Classical relay channel (capacity C_1)



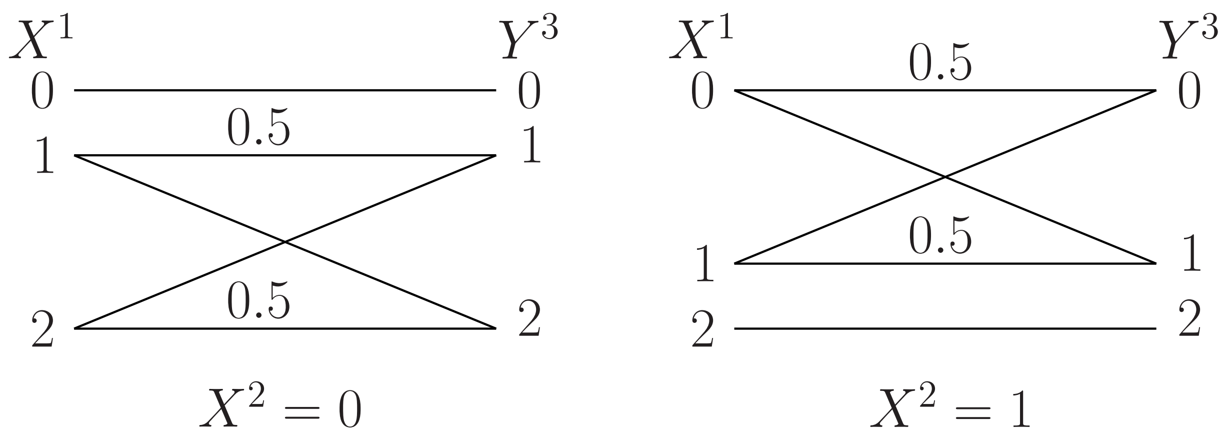
- Relay-without-delay [EH'05] (capacity C_0)



- Is $C_0 = C_1$? (Clearly $C_0 \geq C_1$)
- Does the cut-set bound apply to the relay-without-delay?

Sato 1976 Example

- DM relay channel with $Y^2 = X^1$ and



- Capacity for the classical case $C_1 = 1.161878$ bits/trans [CE'79]
  - Coincides with classical cut-set bound

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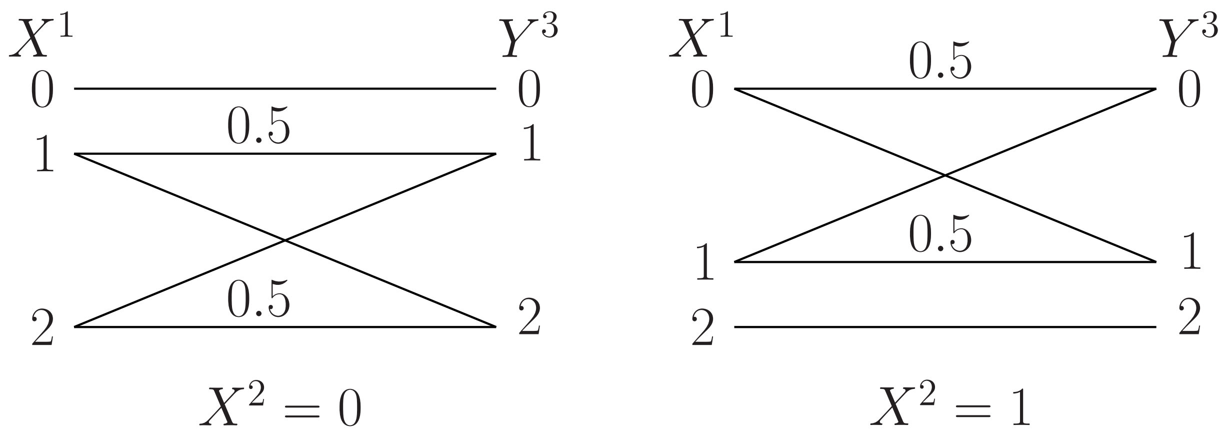


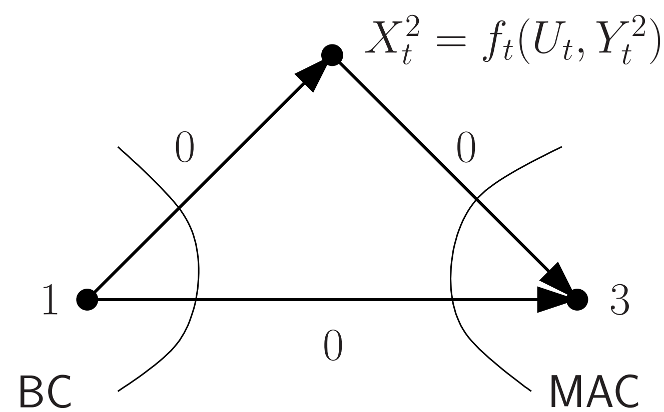
- Capacity for the classical case $C_1 = 1.161878$ bits/trans [CE'79]
  - Coincides with cut-set bound
- Capacity without delay $C_0 = 2(\log 3 - 1) = 1.169925$ bits/trans [EH'05]
  - Achieved using *instantaneous relay coding* (set $X_t^2 = f(Y_t^2)$)
  - $C_0 >$ cut-set bound!!
- Capacity without delay can be $>$ with delay
- Cut-set bound *doesn't* apply to relay-without-delay!!

New Cut-set Bound [EH'05]

- Cut-set bound for relay-without-delay channel:

$$C_0 \leq \max_{p(x^1,u),\, f(u,y^2)} \min\{I(X^1,U;Y^3),\ I(X^1;Y^2,Y^3|U)\},$$

where $x_1 = f(u, y^2)$, and $|\mathcal{U}| \leq \min\{|\mathcal{Y}^3|, |\mathcal{X}^1| \cdot |\mathcal{X}^2|\} + 1$



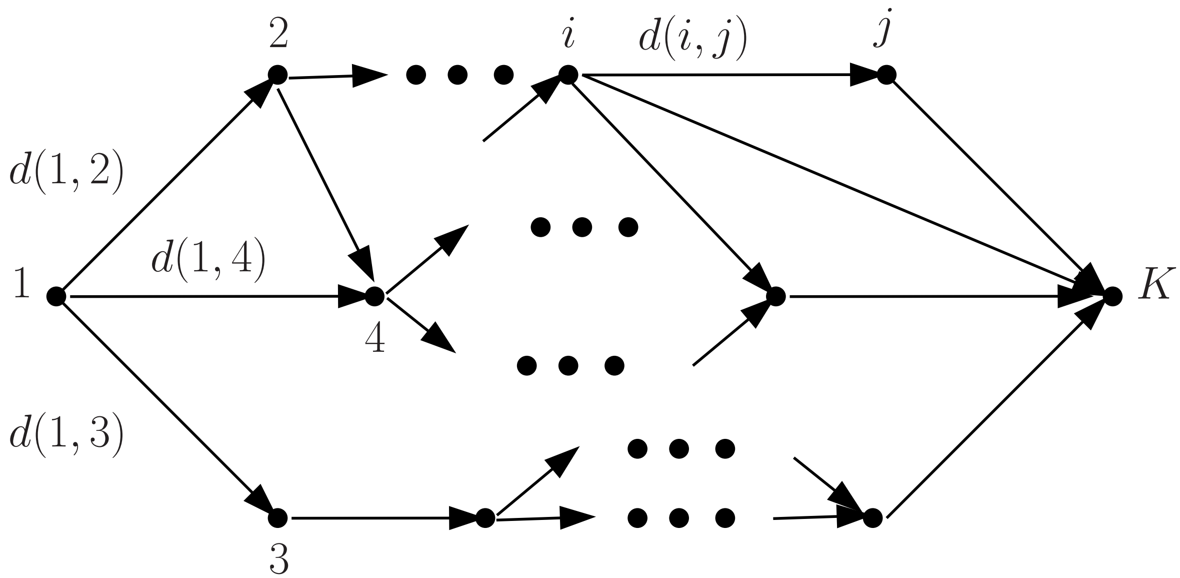
- Proof follows similar lines to cut-set bound:
  - Define: $U_t = Y_{1:t-1}^2$, so $X_t^2 = f_t(U_t, Y_t^2)$
  - Key observation: $(W, Y_{1:t-1}^3) \to (X_t^1, U_t) \to (Y_t^3, Y_t^2)$
- The bound can be strictly larger than cut-set bound (e.g., Sato Ex.)
- The bound reduces to classical cut-set for classical relay

Classical Relay vs. Relay-without-delay

Result	Classical Relay [CE'79]	RWD [EH'05]
Capacity	Not known in general	Not known in general Can be $>$ classical
Upper bound	$\max_{p(x^1,x^2)} \min\{I(X^1,X^2;Y^3),\ I(X^1;Y^3,Y^2\|X^2)\}$ Classical cut-set bound	$\max_{p(x^1,u),\, f(u,y^3)} \min\{I(X^1,U;Y^3),\ I(X^1;Y^2,Y^3\|U)\}$ New cut-set bound $\geq$ Classical
Lower bound	$\max_{p(v,x^1,x^2)} \min\big\{I(X^1,X^2;Y^3),$ $I(X^1;Y^3\|X^2,V)+I(V;Y^2\|X^2)\big\}$ Generalized Block Markov	$\max_{p(u,v,x^1),\, f(u,y^2)} \min\big\{I(X^1,U;Y^3),$ $I(V;Y^2\|U)+I(X^1;Y^3\|U,V)\big\}$ Generalized Block Markov + instant coding
Semi-det capacity $Y^2=g(X^1)$	$\max_{p(x^1,x^2)} \min\{I(X^1,X^2;Y^3),\ I(X^1;Y^2,Y^3\|X^2)\}$ Generalized block Markov	$\max_{p(u,x^1),\, f(u,y^2)} \min\{I(U,X^1;Y^3),\ I(X^1;Y^2,Y^3\|U)\}$ Generalized block Markov + instant coding

DM Relay Network with Delays

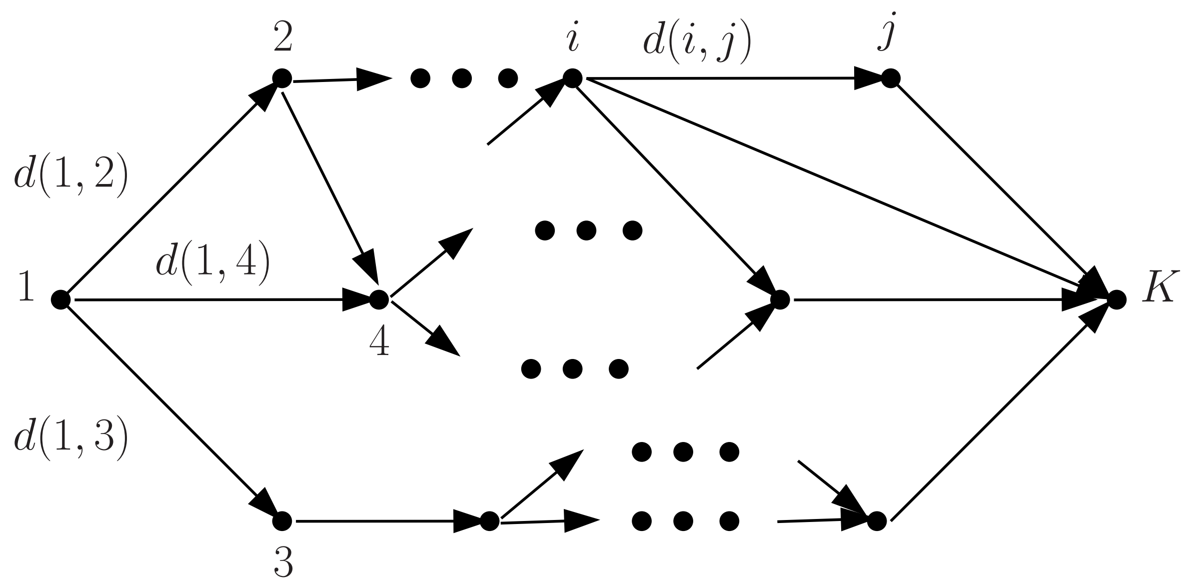
- Weighted directed acyclic graph (DAG) $(\mathcal{N}, \mathcal{E}, d)$:



- X^1 associated with node 1 (sender), Y^K associated with node K (receiver), (X^i, Y^i) associated with node i (relays)
- Family of conditional pmfs: $\left\{p\left(y^i | \{x^j : \ j \in \mathcal{N}_i\}\right), \ i = 2, 3, \ldots, K\right\}$, where $\mathcal{N}_i = \{j \in \mathcal{N} : \ (j, i) \in \mathcal{E}\}$
- Delay $d(i, j) \in \{0, 1, \ldots\}$ assigned to edge $(i, j) \in \mathcal{E}$
- Network is "memoryless"

DM Relay Network with Delays

- Let $d(i)$, $D(i)$ be shortset and longest path delay from 1 to i, respectively



- $(2^{nR}, n)$ code consists of:

  (i) a message $W \in \{1, 2, \ldots, 2^{nR}\}$,

  (ii) encoding function that maps message W into codeword $X^1_{1:n}(w)$,

  (iii) relay encoding functions $X^i_t = f^i_t\left(Y^i_{d(i)+1:t}\right)$, $i = 2, \ldots, K-1$, and

  (iv) decoding function that maps $Y^K_{d(K)+1:D(K)+n}$ into an estimate $\hat{W}$

- Network capacity C is the supremum over the set of achievable rates

Two Results

- How does capacity depend on the link delays?
  - We aleady know that delay matters
- We present two preliminary results:
  1. Sufficient conditions for two networks with the same graph and associated random variables to have the same capacity
  2. New cut-set bound that reduces to classical cut-set for classical relay and to the new cut-set bound (with U) for relay-without-delay

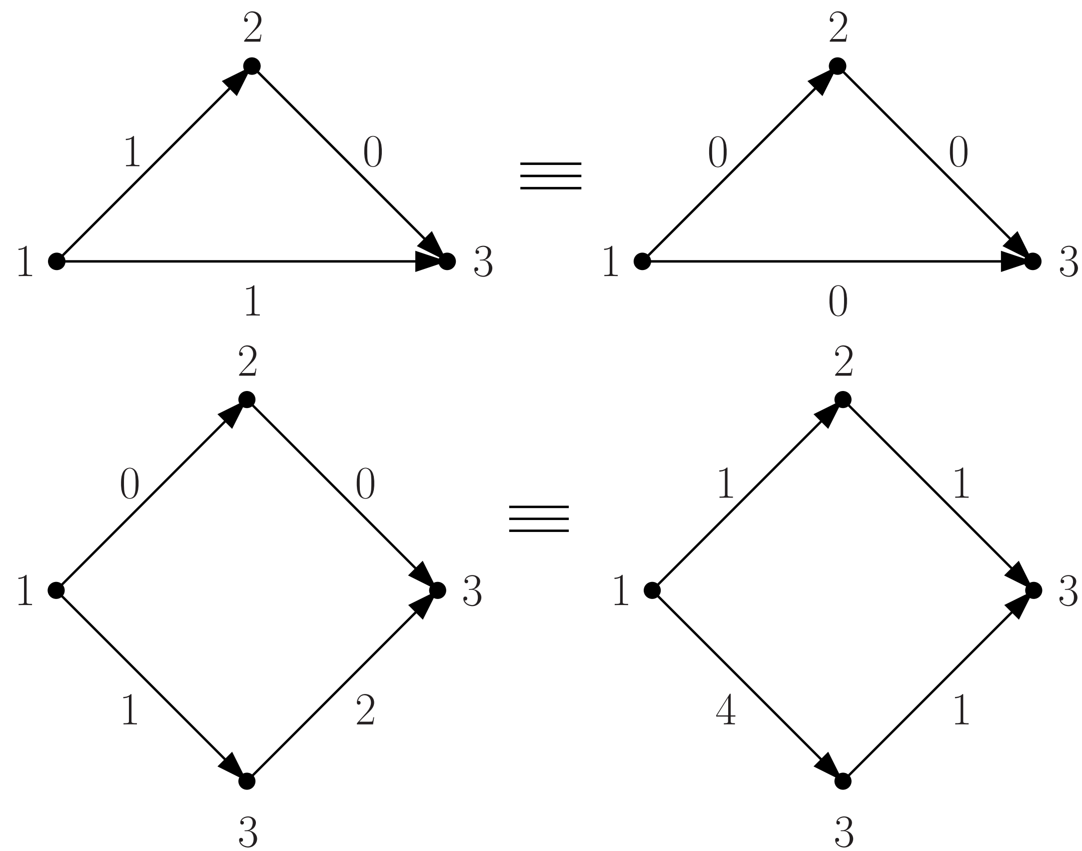
Sufficient Conditions for Equal Capacity

- Consider two relay networks with same DAG, same set of associated random variables, and same conditional pmfs, but *different* link delays

  Let $w_i(p)$ for network $i = 1, 2$ be delay of path p from 1 to K

  Capacities equal if $\forall$ p, $w_1(p) - w_2(p) = m$, for some integer m

- Examples:



Sketch of Proof

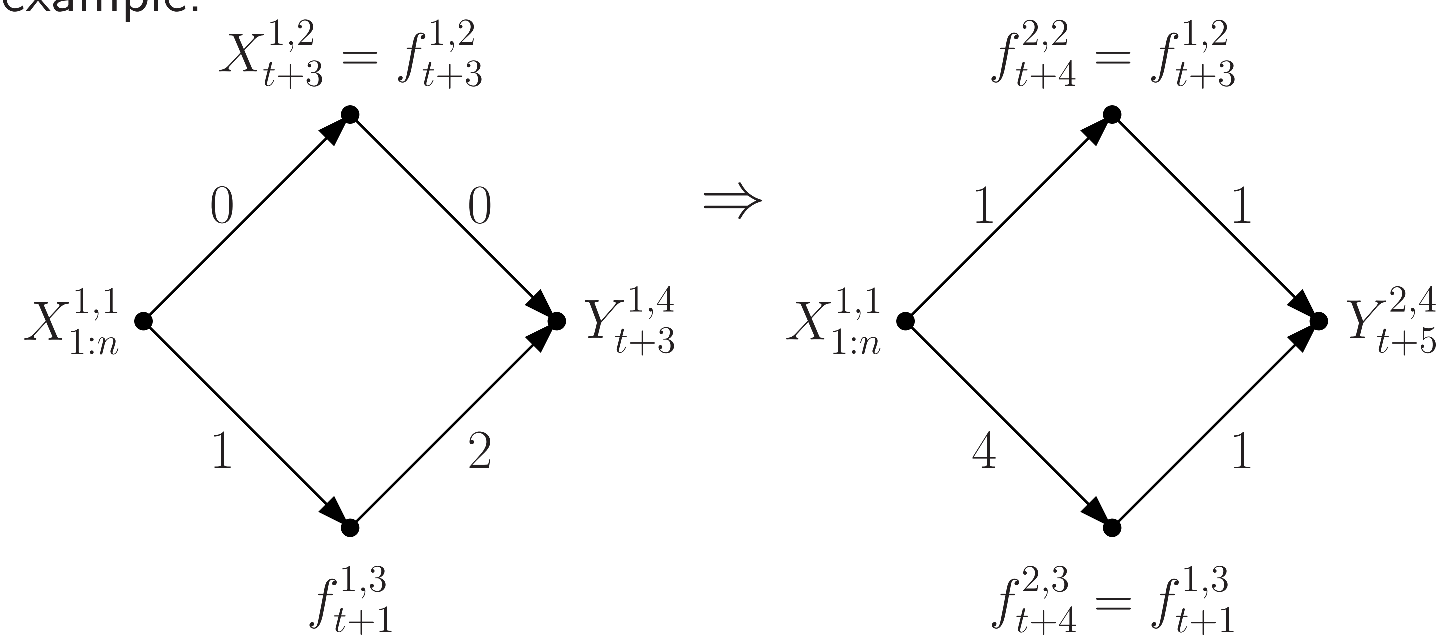
- We show that any $(2^{nR}, n)$ code for one network can be mapped into a code for the other with the same error probability. Code mapping:

  (i) Use same encoding function for node 1

  (ii) Use same relay encoding functions with appropriate time shifts

  (iii) Use same decoding function at node K with appropriate time shift

- For example:



General Cut-Set Bound

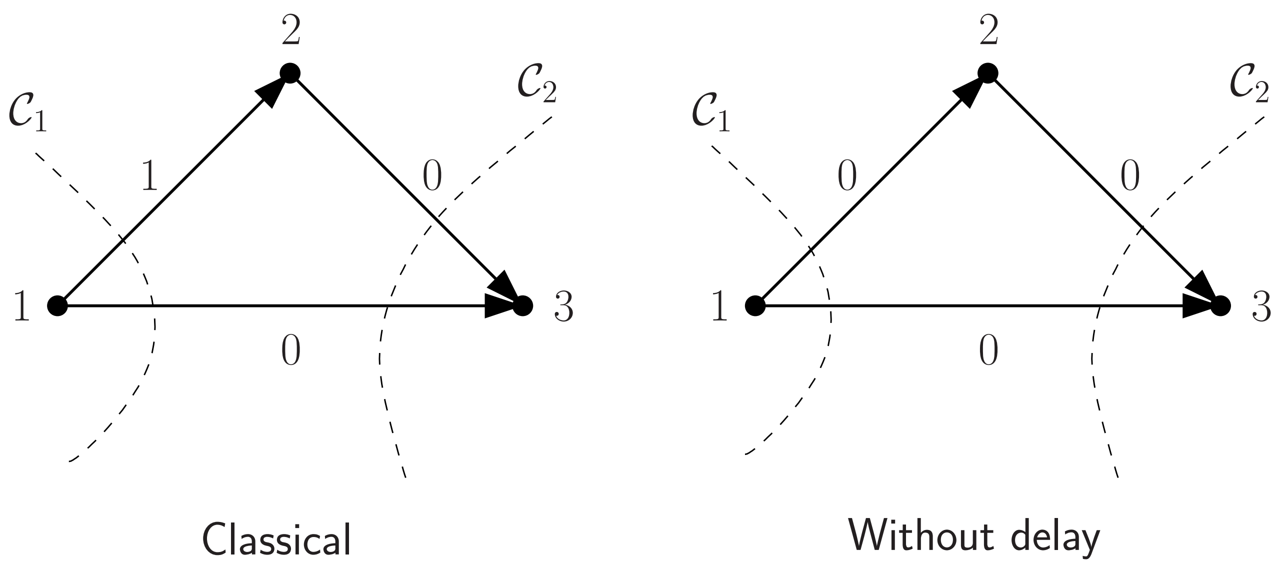
- Upper bound is of the form:

$$C \le \sup \min_{S \subset \mathcal{N}} \{I(X(S), U(S); Y(S^c) | X(S^c), U(S^c)\}$$

- Node 1 in S and node K is in S^c
- Supremum over all joint pmf of r.v.s in $X(S), U(S), X(S^c), U(S^c)$ for all S and over all relay functions involving a U
- The random variables constituting $X(S), U(S), X(S^c), U(S^c)$ are not unique—can write different I expressions for same cut
- The procedure in the paper uses the structure of the network to reduce the number of random variables used
- Bound reduces to cut-set bounds for classical and relay-without-delay
- Two new ingredients beyond classical cut-set:
  - Assigning Us to some nodes instead of Xs (as in RWD bound)
  - Using multiple X r.v.s for same sender

When do We Assign a U?

- Recall classical and relay-without-delay bounds:



$$C_1 \leq \max_{p(x^1,x^2)} \min\left\{I(X^1,X^2;Y^3), I(X^1;Y^3,Y^2|X^2)\right\}$$

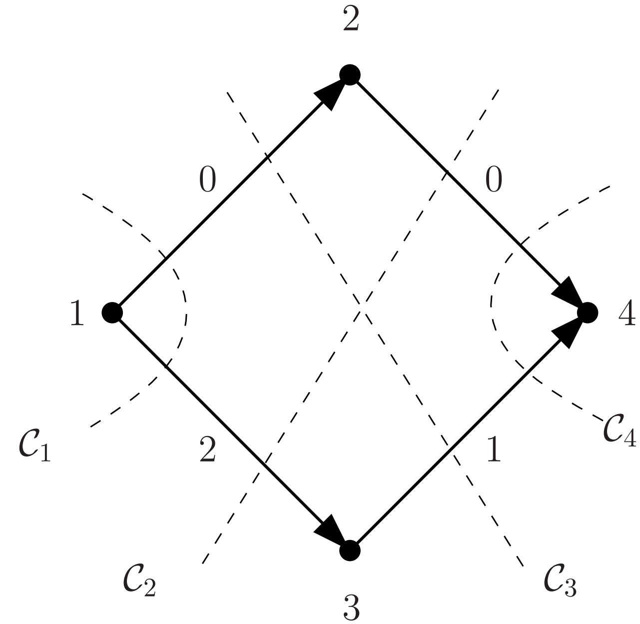
$$C_0 \leq \max_{p(x^1,u),\ f(u,y^2)} \min\{I(X^1,U;Y^3),\ I(X^1;Y^2,Y^3|U)\}$$

- Note that in RWD node 2 is on shortest path from 1 to 3; but not on shortest path in classical

- In general, we assign U^i when node $i \in S$ is on shortest path from node 1 to node $j \in S^c$; otherwise we assign X^i

Example

- 2-relay network:

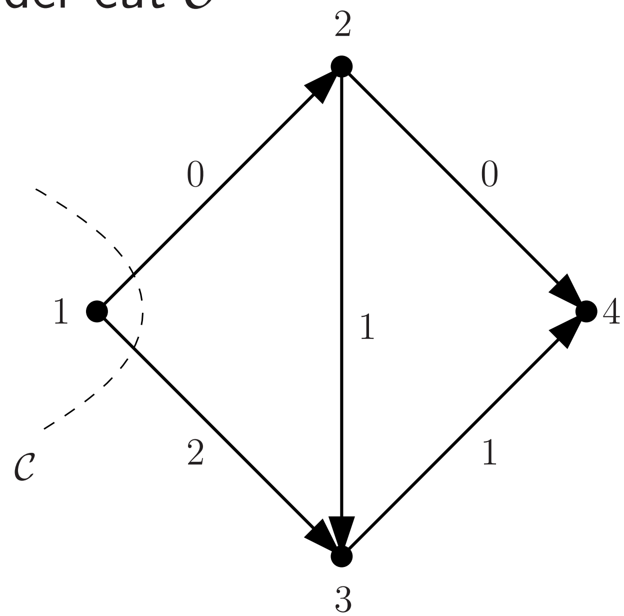


- Relay node 2 is on shortest path, so it is assigned an auxiliary random variable U^2; relay node 3 isn't, so it is assigned X^3
- Cut-set bound is of the form: $C \leq \sup_{p(x^1,u^2,x^3), f(u^2,\ y^2)} \min\{I_1, I_2, I_3, I_4\}$,

$$I_1 = I(X^1; Y^2, Y^3 | U^2, X^3), \qquad I_2 = I(X^1, U^2; Y^3, Y^4 | X^3),$$
$$I_3 = I(X^1, X^3; Y^2, Y^4 | U^2), \qquad I_4 = I(U^2, X^3; Y^4)$$

When Do We Use More RVs for X^i?

- 2-relay network: Consider cut $\mathcal{C}$



- Note that Y^3_{t+1} depends on X^1_{t-1} while Y^2_t depends on X^1_t. There is an edge from 2 to 3, so Y^3_{t+1} also depends indirectly on X^1_t
- Need to include 2 r.v.s for X^1 in the inequality for this cut
- This cut gives the bound

$$C \leq I(X^1_1, X^1_2; Y^2_2, Y^3_3 | U^2_2, X^3_1)$$

Note: Replace X^3_3 with X^3_1 in paper

Conclusion

- Delay can change network information capacity
  - Relative path delay seems to be what matters
- Simplest example is classical relay versus relay-without-delay [EH'05]:
  - No delay makes it possible to perform *instantaneous* relaying
  - Instantaneous relaying can be optimal (Sato Ex.[EH'05])
  - Instantaneous relaying can beat classical cut-set bound (Sato Ex.)
- Two new results for DM relay with delay networks:
  - Sufficient conditions for two networks to have same capacity
  - General cut-set bound for relay networks with delays

    New ingredients:
    - Assign a U when relay node is on shortest path from node 1
    - Assign more than one random variable for same node
- More work needed to better understand the role of delay