Functional Representation of Random Variables and Applications

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Based mostly on joint work with Cheuk Ting Li

Functional representation of random variables

Lemma (see, e.g., EG–Kim (2011))

Given (X, Y), there exists Z independent of X and function $g(x, z)$ such that $Y = g(X, Z)$

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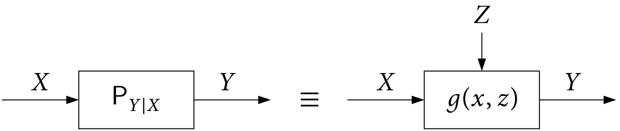

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  - MAC with cribbing encoders (Willems–van der Meulen 1985)
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  - Also see (EG–Kim 2011) for other applications
  - Entropic causal inference (Kocaoglu–Dimakis–Vishwanath–Hassibi 2017)

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- Example: B_1, B_2, B_3, B_4 i.i.d. $\mathrm{Bern}(1/2)$, $X = (B_1, B_2, B_3)$, $Y = (B_2, B_3, B_4)$
  - $Z_1 = B_4$, $\quad Y = (B_2, B_3, Z_1)$ $\quad$ (Z_1 part of Y not in X)

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- What Z is most informative about $Y = g(X, Z)$?

$$H(Y|Z_1) = 2 = I(X;Y), \quad H(Y|Z_2) = H(Y) = 3$$

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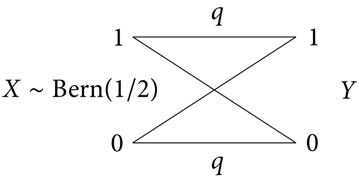
- In general: $H(Y|Z) \geq I(X;Y)$:

$$\begin{aligned} H(Y|Z) &= I(X;Y|Z) \quad (Y = g(X,Z)) \\ &= I(X;Y,Z) \quad (X \text{ and } Z \text{ independent}) \\ &\geq I(X;Y) \end{aligned}$$

Example (doubly symmetric binary r.v.s)



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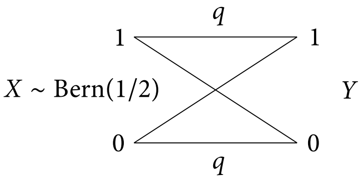

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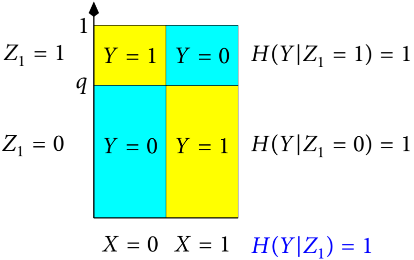

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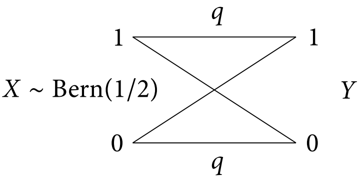

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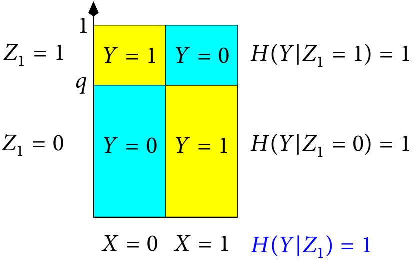
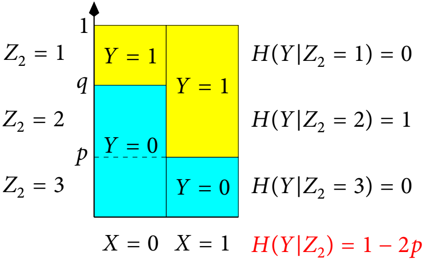
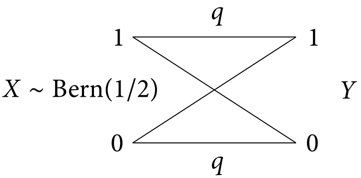
- Let $Z_2 = 1, 2, 3$ w.p. $p, 1 - 2p, p$, respectively, indep. of X

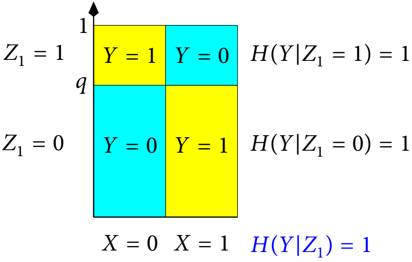


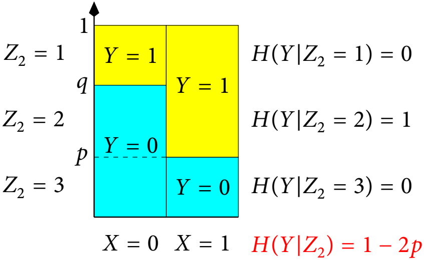




$q = 1 - p, \; p \in (0, 0.5)$

- Can show: $\min_{Z,g} H(Y|Z) = 1 - 2p$, i.e., second construction is optimal
- But $1 - 2p > 1 - H(p) = I(X;Y)$ (cannot always achieve I lower bound)





General upper bound on $H(Y|Z)$

Strong functional representation lemma (SFRL) (Li–EG 2018)

Given (X, Y), there exists Z independent of X and function $g(x, z)$ such that $Y = g(X, Z)$, and

$$I(X; Y) \leq H(Y|Z) < I(X; Y) + \log(I(X; Y) + 1) + 4$$

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- Provides simple achievability results for several coding setups

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- Upper bound can be quite loose, e.g., for binary example with $p = 0.11$,
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- For $(X, Y) = (X^n, Y^n)$ i.i.d.: $(1/n)H(Y^n|Z_n) \leq I(X; Y) + O(\log n/n) \approx I(X; Y)$

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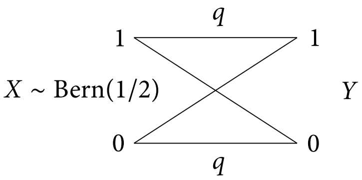
- There are examples where log term is necessary, SFRL tight within 5 bits

- Recall optimal Z_2 construction for example



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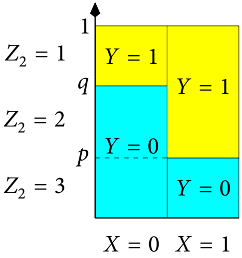
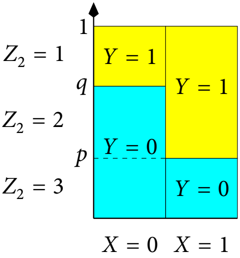
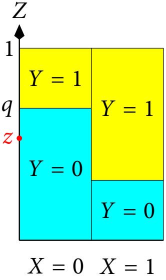


Back to doubly symmetric binary r.v.s example

- Recall optimal Z_2 construction for example
- Can equivalently let $Z \sim \mathrm{Unif}[0,1]$, and:

  For $X = 0$, set $y = 0$ if $\dfrac{z}{q} \le \dfrac{1-z}{p}$; for $X = 1$, set $y = 0$ if $\dfrac{z}{p} \le \dfrac{1-z}{q}$





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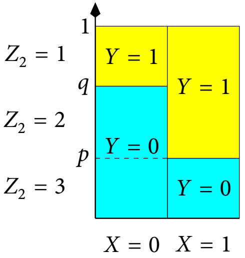
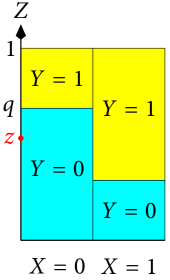
- In general for $|\mathcal{Y}| = 2$, optimal construction is $Z \sim \mathrm{Unif}[0,1]$ and:

$$y = g(x,z) = \operatorname{argmin}\left\{ \frac{z}{p_{Y|X}(0|x)}, \frac{1-z}{p_{Y|X}(1|x)} \right\}$$





Exponential construction of Z, g

- Let $\mathcal{Y} = \{1, 2, \ldots, l\}$
- Take $Z = (Z_1, Z_2, \ldots, Z_l)$ i.i.d. $\mathrm{Exp}(1)$ r.v.s independent of X, set

$$y = g(x, z) = \operatorname*{argmin}_{y'} \frac{Z_{y'}}{p_{Y|X}(y'|x)}$$

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$$\mathsf{P}\{\operatorname*{argmin}_{y'} \mathrm{Exp}(p_{Y|X}(y'|x)) = y\} = p_{Y|X}(y|x) \Rightarrow g(x, Z) \sim p_{Y|X}(.|x)$$

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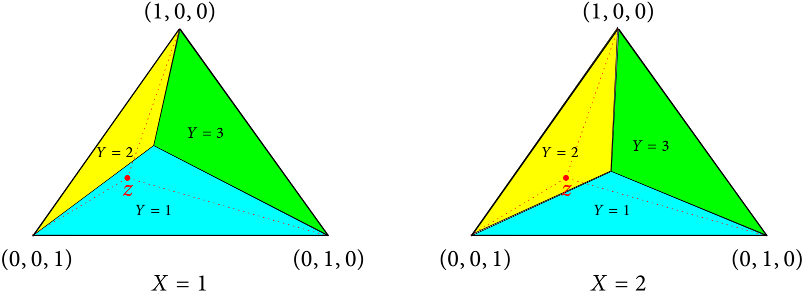
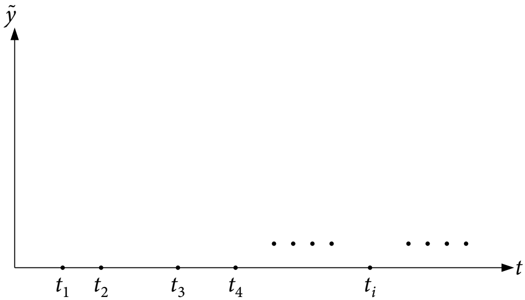
- That is, pick Z uniform over probability simplex in $\mathbb{R}^{|\mathcal{Y}|-1}$



Poisson construction for general Y

- Let $T_1, T_2, \dots$ be arrival times of Poisson process with $\lambda = 1$, indep. of X



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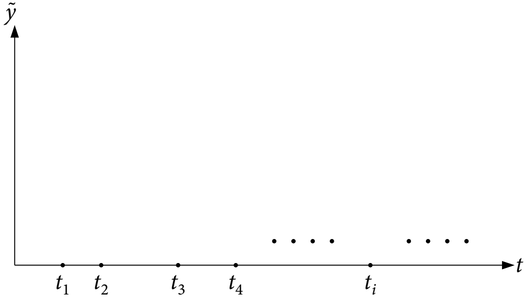

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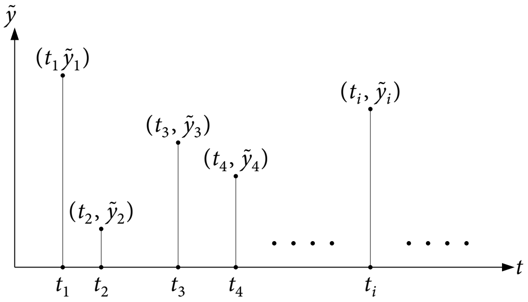

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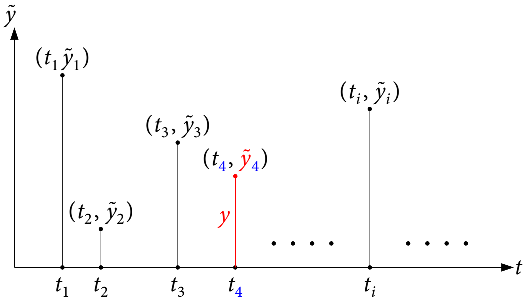

Example

- Let $Y \sim \mathrm{Unif}[0,1]$, $Y|\{X=x\} \sim \mathrm{Unif}[x, 1-x]$, $x \in [0, 1/2]$, hence

$$\begin{aligned} k(x,z) &= \operatorname*{argmin}_i t_i \cdot \frac{f_Y(\tilde{y}_i)}{f_{Y|X}(\tilde{y}_i|x)} \text{ for } \tilde{y}_i \in [x, 1-x] \\ &= \operatorname*{argmin}_i t_i \cdot (1-2x) \text{ for } \tilde{y}_i \in [x, 1-x] \end{aligned}$$

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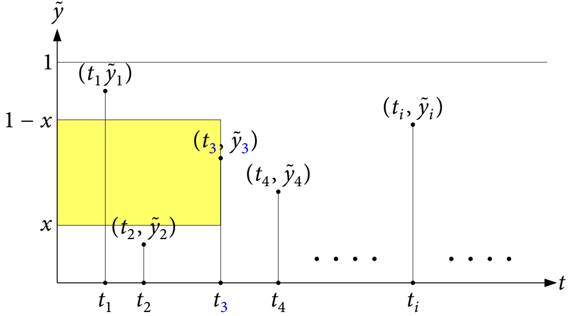

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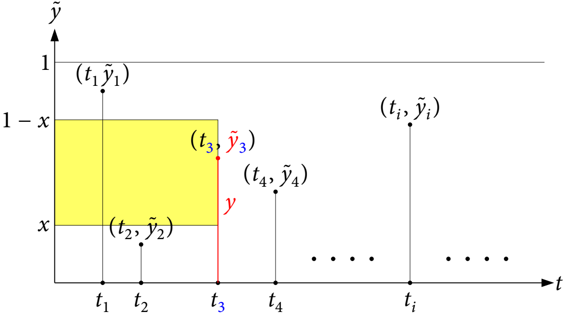
$$y = \tilde{y}_k$$



SFRL proof outline

Strong functional representation lemma (Li–EG 2018)

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- Poisson construction: $Z = \{(T_i, \tilde{Y}_i)\}$ marked PP with intensity measure $\mu \times \mathsf{P}_Y$,

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  By the mapping theorem, $\{(T_i \cdot d\,\mathsf{P}_Y / d\,\mathsf{P}_{Y|X}(\cdot|x)(\tilde{Y}_i), \tilde{Y}_i)\}$ is PP with intensity measure $\mu \times \mathsf{P}_{Y|X}$

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$$Y = g(x, Z) = \tilde{Y}_{k(x,Z)}, \text{ where } k(x, Z) = \operatorname*{argmin}_i T_i \cdot \frac{d\,\mathsf{P}_Y}{d\,\mathsf{P}_{Y|X}(\cdot|x)}(\tilde{Y}_i)$$

- $g(x, Z) \sim \mathsf{P}_{Y|X}(\cdot|x)$: Consider mapping $(t, y) \mapsto (t \cdot d\,\mathsf{P}_Y / d\,\mathsf{P}_{Y|X}(\cdot|x)(y), y)$

  By the mapping theorem, $\{(T_i \cdot d\,\mathsf{P}_Y / d\,\mathsf{P}_{Y|X}(\cdot|x)(\tilde{Y}_i), \tilde{Y}_i)\}$ is PP with intensity measure $\mu \times \mathsf{P}_{Y|X}$

  Hence, $\Theta = \min_i T_i \cdot \frac{d\,\mathsf{P}_Y}{d\,\mathsf{P}_{Y|X}(\cdot|x)}(\tilde{Y}_i) \sim \mathrm{Exp}(1)$, $\tilde{Y}_K | \{X = x\} \sim \mathsf{P}_{Y|X}(\cdot|x)$

SFRL proof outline

- Since Y is a function of Z and K: $H(Y|Z) \leq H(K)$
- Proposition (max. $H(K)$ for fixed $\mathsf{E}(\log K)$): Let $K \in \mathbb{N}$, then

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  It's not difficult to show: $\lambda(\tilde{y}) \le \theta \cdot d\,\mathsf{P}_{Y|X}(\cdot|x)/d\,\mathsf{P}_Y(\tilde{y})$

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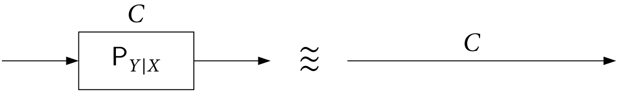
- Taking expect. over X and substituting into Proposition complete proof

Applications of SFRL

- Upper bound on rate of one-shot (exact) channel simulation
- One-shot lossy compression
- Minimax learning for distributed inference (Li–Wu–Özgür–EG 2018)

Background on channel simulation

- Shannon (1948) channel capacity theorem can be interpreted as:
  DMC with capacity C can simulate noiseless channel with capacity C



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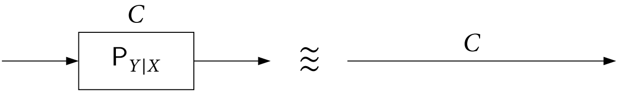
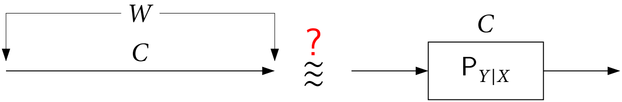


- Bennett–Shor–Smolin–Thapliyal (2002) asked the reverse question:
  Can noiseless channel with capacity C and common randomness simulate any DMC with capacity C?



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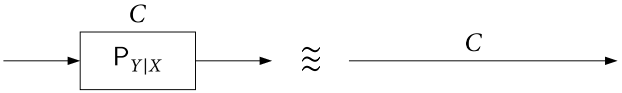

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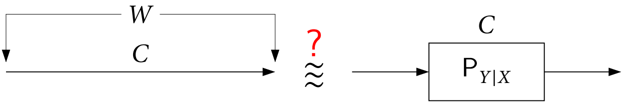
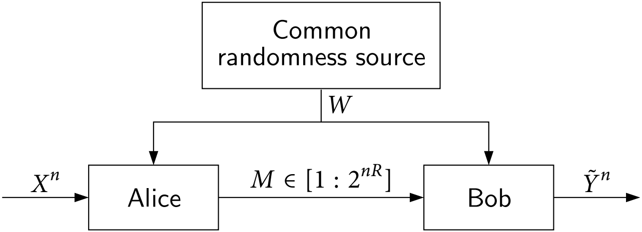


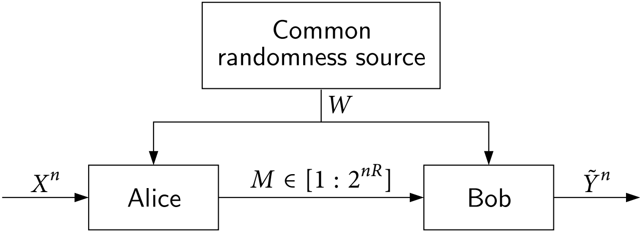
- Their motivation was to answer this question for entanglement-assisted quantum channels

Approximate channel simulation



- W unlimited common randomness; X arbitrary process; $p(y|x)$ DMC
- Alice maps every (x^n, w) pair into an index $m(x^n, w) \in [1 : 2^{nR}]$
- Bob generates $\tilde{Y}^n(m(x^n, W), W) \sim q(y^n|x^n)$

Approximate channel simulation



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- Bob generates $\tilde{Y}^n(m(x^n, W), W) \sim q(y^n|x^n)$
- R is achievable if there exists sequence of simulation schemes such that

$$\lim_{n\to\infty} \left\| p(x^n)q(y^n|x^n) - p(x^n)\prod_{i=1}^{n} p_{Y|X}(y_i|x_i) \right\|_{\mathrm{TV}} = 0$$

- Optimal (approx.) simulation rate $R^*_{\mathrm{ch-sim}}$ is inf. over achievable rates

Approximate channel simulation

Theorem (Bennett–Shor–Smolin–Thapliyal 2002)

$$R^*_{\text{ch-sim}} = \max_{p(x)} I(X;Y) \quad \text{(capacity of DMC)}$$

- Hence reverse Shannon channel capacity theorem holds for DMC

Approximate channel simulation

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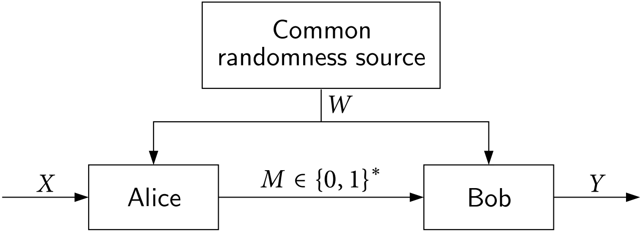
$$R^*_{\mathrm{ch-sim}} = \max_{p(x)} I(X;Y) \quad \text{(capacity of DMC)}$$

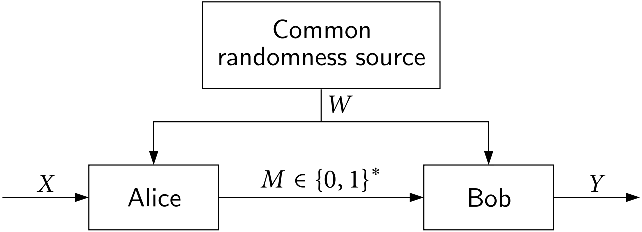
- Hence reverse Shannon channel capacity theorem holds for DMC
- They also established partial results for certain quantum channels
- Follow on work (Cuff 2013, Bennett–Devetak–Harrow–Shor–Winter 2014)

One-shot exact channel simulation



- W unlimited common randomness; $X \sim \mathsf{P}_X$, $\mathsf{P}_{Y|X}$ given channel

One-shot exact channel simulation



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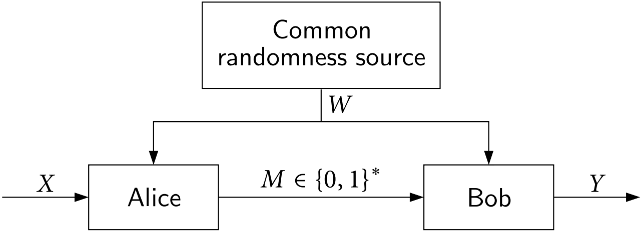
- For each w, Alice uses prefix code to map each x into $m(x, w) \in \{0,1\}^*$
- Bob generates $Y = y(m(x, W), W) \sim \mathsf{P}_{Y|X}(.|x)$
- Let L be the length of the index M

One-shot exact channel simulation



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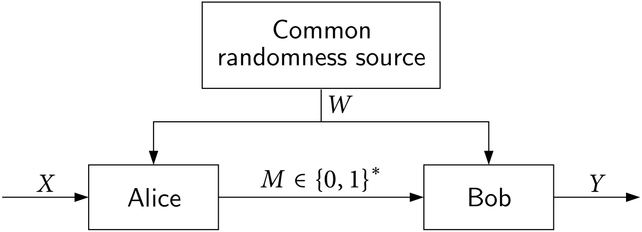
- Let L be the length of the index M
- Optimal average simulation rate is $\bar{R}^*_{\text{ch-cim}} = \inf_{\text{generators}} \mathsf{E}(L)$

One-shot exact channel simulation



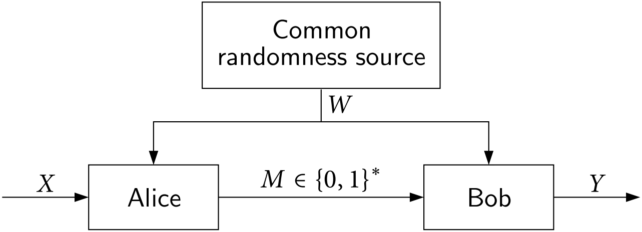
Theorem (Harsha–Jain–McAllester–Radhakrishnan 2010)

For discrete $X \sim p(x)$, and DMC $p(y|x)$,

$$I(X;Y) \le \bar{R}^*_{\text{ch}-\text{sim}} \le I(X;Y) + (1+\epsilon)\log(I(X;Y)+1) + c_\epsilon$$

- More generally they showed for any x: $\bar{R}^*_{\text{ch}-\text{sim}} \le C + (1+\epsilon)\log(C+1) + c_\epsilon$
- Proof uses rejection sampling and is quite involved

One-shot exact channel simulation



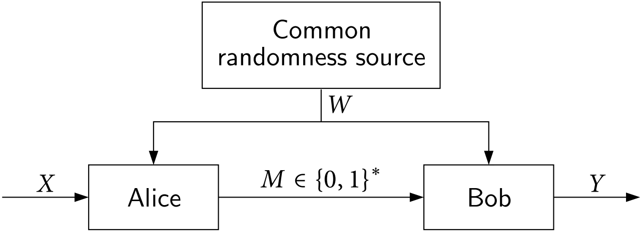
Theorem (Li–EG 2018)

For $X \sim \mathrm{P}_X$, and general memoryless channel $\mathrm{P}_{Y|X}$,

$$I(X;Y) \le \bar{R}^*_{\mathrm{ch-sim}} < I(X;Y) + \log(I(X;Y)+1) + 5$$

- Proof of upper bound uses SFRL
- Can be extended to arbitrary x case

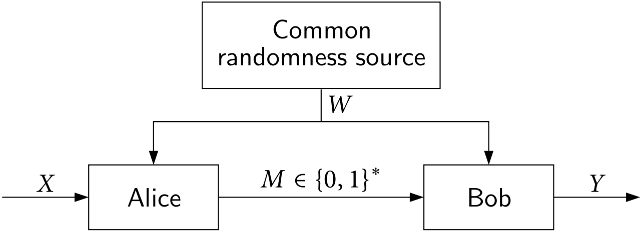
Proof of upper bound using SFRL



- By SFRL, there exists W indep. of X such that $Y = g(X, W)$, and

$$H(Y|W) < I(X;Y) + \log(I(X;Y) + 1) + 4$$

Proof of upper bound using SFRL



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- For each w, map $y(x, w)$ into $m(x, w)$ using Huffman code for $p_{Y|W}(y|w)$
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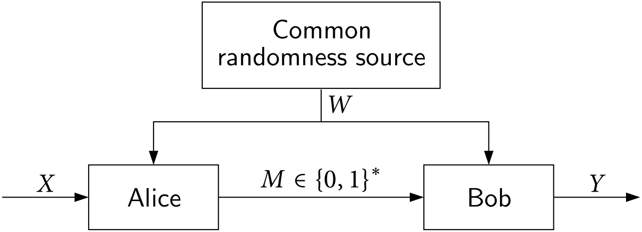

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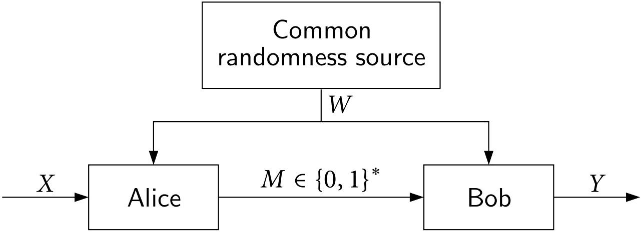

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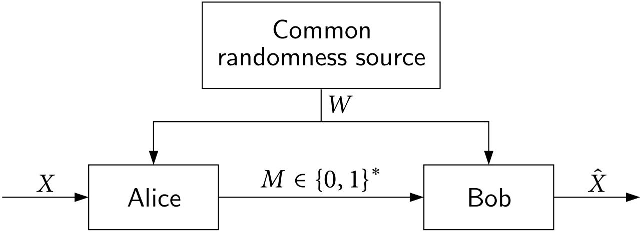
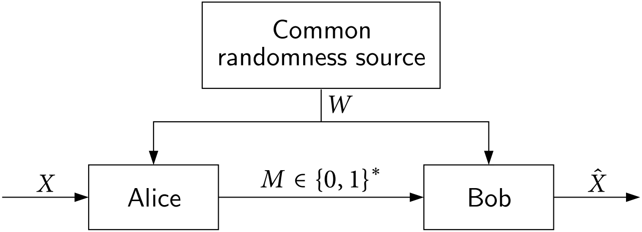
- Codes $\{y(x,w), m(x,w)\}$ are provided to Alice and Bob
- Given (x,w), Alice computes $y(x,w)$; given (m,w), Bob recovers y
- Hence, $\bar{R}^*_{\text{ch-sim}} \le \mathsf{E}(L) < H(Y|W) + 1 < I(X;Y) + \log(I(X;Y)+1) + 5$

One-shot lossy source coding



- $X \sim \mathsf{P}_X$, $\hat{\mathcal{X}}$ reproduction alphabet, $d(x, \hat{x}) \geq 0$ distortion measure

One-shot lossy source coding



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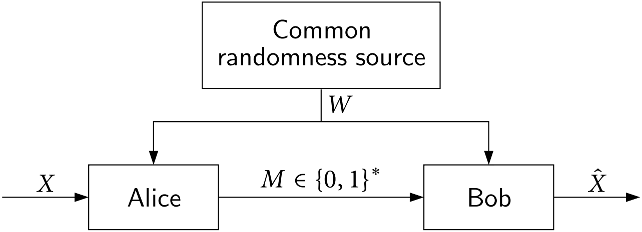

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- $(\bar{R}, D)$ is achievable if there exits code with $\bar{R} = \mathsf{E}(L)$, $\mathsf{E}(d(X, \hat{X})) \leq D$
- Avg rate-dist. function $\bar{R}(D)$ is inf over all achievable $\bar{R}$: $\mathsf{E}(d(X, \hat{X})) \leq D$

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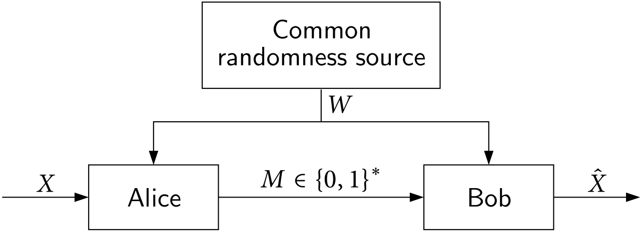

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Theorem (Li–EG 2018)

$$R(D) \le \bar{R}(D) < R(D) + \log(R(D)+1) + 5,$$

where $R(D) = \inf_{P_{\hat{X}|X}:\, \mathsf{E}(d(X,\hat{X})) \le D} I(X;\hat{X})$ (rate-dist. function for asymptotic case)

Proof of upper bound using SFRL

- Let $\hat{X}$ attain $I(X;\hat{X}) = R(D)$ and $\mathsf{E}(d(X,\hat{X})) \le D$

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$$H(\hat{X}|W) < I(X;\hat{X}) + \log(I(X;\hat{X}) + 1) + 4 = R(D) + \log(R(D) + 1) + 4 \ (*)$$

- By Carathéadory's theorem, $|\mathcal{W}| \le 2$ suffices to satisfy $(*)$, $\mathsf{E}(d(X,\hat{X})) \le D$

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- Hence, $\bar{R}(D) \le \mathsf{E}(L) < H(\hat{X}|W) + 1 < R(D) + \log(R(D)+1) + 5,\ \mathsf{E}(d(X,\hat{X})) \le D$

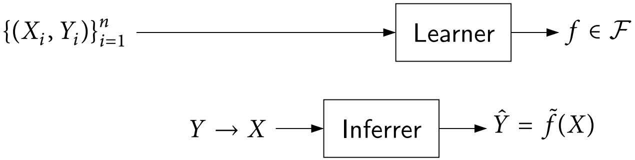
Related work

- Pinkston (1967) studied variable-length finite blocklength lossy compression for i.i.d. source, per-letter distortion
- Zhang–Yang–Wei (1997) established similar order bound to ours for finite blocklength
- Kostina–Polyanskiy–Verdú (2015) studied variable length finite blocklength lossy compression with prob. of distortion constraint
- Our coding scheme resembles Song–Cuff–Poor (2016) likelihood encoder

Applications of SFRL

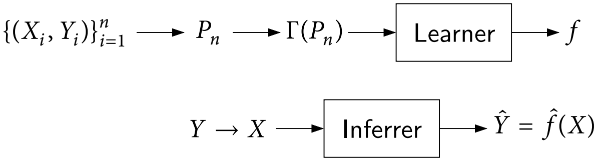
- Upper bound on rate of one-shot (exact) channel simulation
- One-shot lossy compression
- Minimax learning for distributed inference (Li–Wu–Özgür–EG 2018)

Supervised learning



- Risk function: $l(y, \hat{y})$, P_n empirical pmf of (X, Y), function class $\mathcal{F}$
- Empirical risk minimization: choose $\tilde{f} = \operatorname*{argmin}_{f \in \mathcal{F}} \mathsf{E}_{P_n}(l(Y, \hat{Y}))$

Minimax learning



- Minimax learning: choose $\hat{f} = \operatorname{argmin}_f \max_{P \in \Gamma(P_n)} \mathsf{E}_P(l(Y, \hat{Y}))$

  $\Gamma(P_n)$: ambiguity set around P_n, e.g.,
  - Set of pmfs with same 1st, 2nd moments as P_n (Farnia–Tse 2016)
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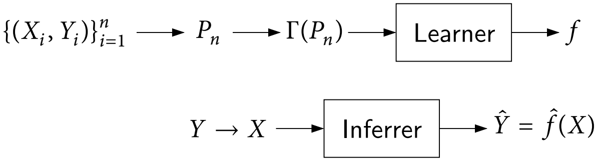

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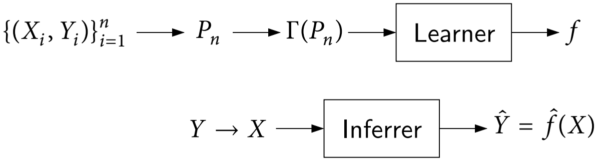

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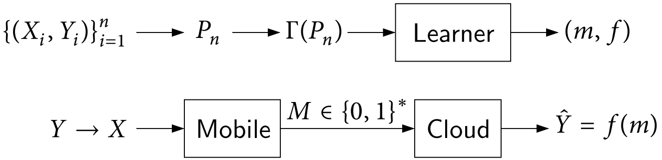
  - Recovers linear/logistic regression for suitable l, Γ

Minimax learning for distributed inference



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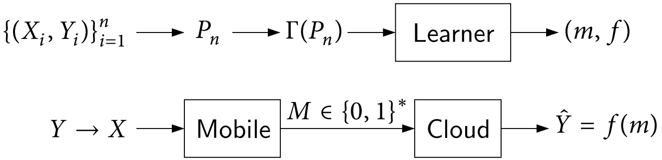

- Assume common randomness W available between cloud/mobile
- Mobile maps every (x, w) into index $m(x, w)$
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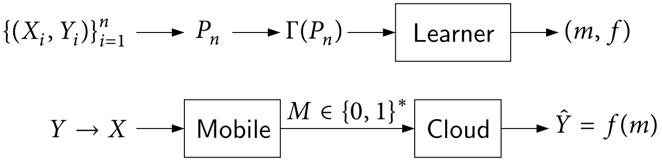

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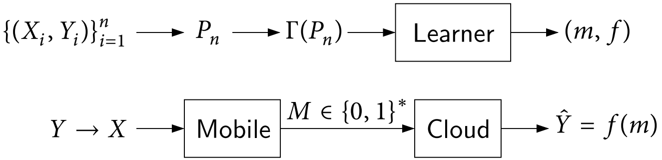

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Theorem (Li–Wu–Özgür–EG 2018)

Let Γ be convex, then

$$L_\lambda^* \geq \inf_{\hat{P}_{\hat{Y}|X}} \sup_{P\in\Gamma} \left[\mathsf{E}_P(l(Y,\hat{Y})) + \lambda I(X;\hat{Y}) \right]$$

$$L_\lambda^* < \inf_{\hat{P}_{\hat{Y}|X}} \sup_{P\in\Gamma} \left[\mathsf{E}_P(l(Y,\hat{Y})) + \lambda (I(X;\hat{Y}) + 2\log(I(X;\hat{Y})+1) + 6) \right]$$

Proof outline of upper bound

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- For $\Gamma = \{P\}$, problem reduces to one-shot noisy lossy compression

  Proof essentially same as for one-shot lossy compression via SFRL,

$$L_\lambda^* < \inf_{\hat{P}_{\hat{Y}|X}} \left[\mathsf{E}(l(Y,\hat{Y})) + \lambda(I(X;\hat{Y}) + \log(I(X;\hat{Y})+1) + 5)\right]$$

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- For general Γ, we need refined version of SFRL:

  For $\mathsf{P}_{\hat{Y}|X}$, $\tilde{\mathsf{P}}_{\hat{Y}}$, there exists r.v. W, two functions $k(x,w)\in\mathbb{N}$, $\hat{y}(k,w)$:

$$\hat{y}(k(x,W),W) \sim \mathsf{P}_{\hat{Y}|X}$$
$$\mathsf{E}(\log k(x,W)) \le D(\mathsf{P}_{\hat{Y}|X}(.|x)||\tilde{\mathsf{P}}_{\hat{Y}}) + 1.6$$

- Encode K using Elias (1975) codes: $\mathsf{E}(T) \le \mathsf{E}(\log K) + 2\log(\mathsf{E}(\log K)+1)+1$
- Rest of proof is technical, see details in (Li–Wu–Özgür–EG 2018)

Principle of max risk-information cost

- Minimax risk-rate cost: $L_\lambda^* = \inf_{f,m} \sup_{P \in \Gamma} \big(\mathsf{E}_p(l(Y, \hat{Y})) + \lambda\, \mathsf{E}_p(T) \big)$

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- To design robust descriptor-estimator pair that works for every $p \in \Gamma$,
  - First find: $p^* = \operatorname{argmax}_{p\in\Gamma} \min_{p_{\hat{Y}|X}} \big(\mathsf{E}(l(Y,\hat{Y})) + \lambda I(X;\hat{Y}) \big)$
  - Then find: $p^*_{\hat{Y}|X} = \operatorname{argmin}_{p_{\hat{Y}|X}} \big(\mathsf{E}_{p^*}(l(Y,\hat{Y}) + \lambda I_{p^*}(X;\hat{Y}) \big)$
- Extends maximum conditional entropy principle in (Farnia–Tse 2016)

Linear regression

- Let $\mathbf{X} \in \mathbb{R}^d$, $Y, \hat{Y} \in \mathbb{R}$, $l(y, \hat{y}) = (y - \hat{y})^2$, $\mathsf{E}(\mathbf{X}) = \mathbf{0}$, $\mathsf{E}(Y) = 0$

$$\Gamma = \{P_{\mathbf{X},Y} : \ \mathsf{E}(\mathbf{X}) = \mathbf{0}, \ \mathsf{E}(Y) = 0, \ \Sigma_{\mathbf{X}}, \ C_{\mathbf{X}Y}, \ \text{same as } P_n\}$$

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- Minimax solution: $P^*_{\mathbf{X},Y}$ Gaussian with same mean, covariance as P_n,

$$\hat{Y} = \begin{cases} a \cdot C^t_{\mathbf{X}Y}\Sigma^{-1}_{\mathbf{X}}\mathbf{X} + Z & \text{if } a > 0, \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{L}^*_\lambda = \begin{cases} \sigma^2_Y - C^t_{\mathbf{X}Y}\Sigma^{-1}_{\mathbf{X}}C_{\mathbf{X}Y} - \frac{\lambda}{2}\log e(1-a) & \text{if } a > 0, \\ \sigma^2_Y & \text{otherwise,} \end{cases}$$

$$a = 1 - \frac{\lambda \log e}{2C^t_{\mathbf{X}Y}\Sigma^{-1}_{\mathbf{X}}C_{\mathbf{X}Y}}, \quad Z \sim \mathrm{N}\big(0, a\lambda \log e/2\big) \text{ independent of } \mathbf{X}$$

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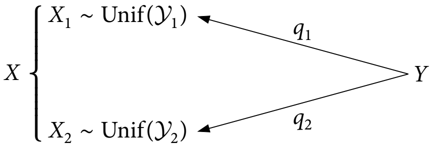
- $\lambda = 0$ (no rate constraint) $\Rightarrow$ linear regression (Farnia–Tse 2016)
- Straightforward estimate-compress scheme optimal:
  - Estimate: Compute MMSE estimate of Y given $\mathbf{X}$
  - Compress: Scale MMSE estimate and add Z to obtain $\hat{Y}$

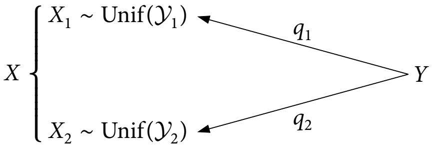


$|\mathcal{Y}_i| = k_i,\ i = 1, 2,\ q_2 = 1 - q_1$

- Let $\mathcal{Y} = \hat{\mathcal{Y}} = \mathcal{Y}_1 \cup \mathcal{Y}_2$, $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \emptyset$, $|\mathcal{Y}_1| = k_1$, $|\mathcal{Y}_2| = k_2$; $l(y, \hat{y}) = \mathbb{1}_{\{\hat{y} \neq y\}}$;

  $\Gamma = \{P\}$, P: $X = (X_1, X_2) \sim \mathrm{Unif}[\mathcal{Y}_1 \times \mathcal{Y}_2]$, $Y = X_1$ w.p. q_1 or X_2 w.p. $q_2 = 1 - q_1$

Classification example



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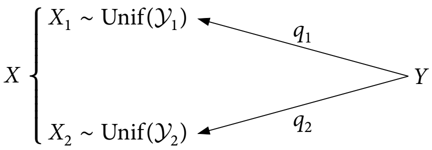

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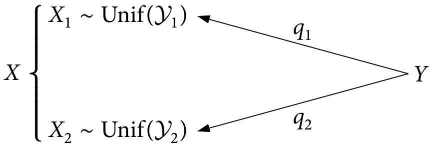

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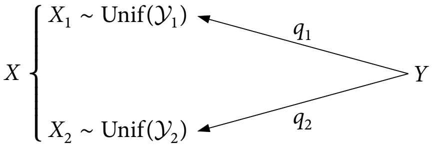

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    $$\bar{L}_\lambda = 1 - \lambda \log \max\{a_1/k_1, 2^{\lambda^{-1} q_2 k_2^{-1}}\} \ (**)$$

  - If $k_1 \gg k_2$, optimal scheme is to pick random $y \in \mathcal{Y}_2$ and $(**)$ can be >> $(*)$

Summary

- Strong functional representation lemma (SFRL)
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  - Other applications:

    Multiple description coding, Gray–Wyner system, Gelfand–Pinsker

Thank you!

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