Communication Cost of Distributed Computing

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Motivation

- Performance of distributed information processing systems often limited by communication
  - Digital VLSI
  - Multi-processors
  - Data centers
  - Peer-peer networks
  - Sensor networks
  - Networked mobile agents

- Purpose of communication is to make decision, compute function, coordinate action based on distributed data

  How much communication is needed to perform such a task?
Wireless Video Camera Network

- **Today's surveillance systems**: Analog; costly; human operated
- **Future systems**: Digital; networked; self-configuring; automated detection, e.g., suspicious activity, localization, tracking, ...

Sending all video data requires large **communication bandwidth**, energy
Distributed Computing Approach (YSEG 04, EEG 07)

- Clusterhead
- Local processing:
  - Subtract background
  - Scan-lines
- Scan-lines
Experimental Setup

View of the setup

View from a camera

Top view of room

Scan-lines from 16 cameras
Problem Setup

- Network with \( m \) nodes. Node \( j \) has data \( x_j \)
- Node \( j \) wishes to estimate \( g_j(x_1, x_2, \ldots, x_m), j = 1, 2, \ldots, m \)
- Nodes communicate and perform local computing

What is the minimum amount of communication needed?
Problem formulated and studied in several fields:

- **Computer science**: Communication complexity; gossip
- **Information theory**: Coding for computing; CEO problem
- **Control**: Distributed consensus

Formulations differ in:

- Data model
- Type of communication-local computing protocol
- Estimation criterion
- Metric for communication cost
Outline

1. Communication Complexity
2. Information Theory: Lossless Computing
3. Information Theory: Lossy Computing
4. Distributed Consensus
5. Distributed Lossy Averaging
6. Conclusion
Communication Complexity (Yao 79)

- Two nodes, two-way communication
- Node \( j = 1, 2 \) has binary \( k \)-vector \( x_j \)
- Both nodes wish to compute \( g(x_1, x_2) \)
- Use round robin communication and local computing protocol

What is minimum number of bits of communication needed (communication complexity, \( C(g) \))?
Example (Equality Function)

\[ g(x_1, x_2) = 1 \text{ if } x_1 = x_2, \text{ and } 0, \text{ otherwise} \]

**Upper bound:** \( C(g) \leq k + 1 \) bits

For \( k = 2 \): \( C(g) \leq 3 \) bits

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**Lower bound:** *Every* communication protocol partitions \( \{0, 1\}^k \times \{0, 1\}^k \) into \( g \)-monochromatic rectangles, each having distinct code. If \( r(g) \) is minimum number of such rectangles, then \( C(g) \geq \log r(g) \)

For \( k = 2 \), \( r(g) = 8 \Rightarrow C(g) \geq 3 \) bits

In general, \( C(g) \geq k + 1 \Rightarrow C(g) = k + 1 \) bits
**Number-on-Forehead Game**

- $m$ players. Player $j$ has binary $k$-vector $x_j$ on her forehead.
- Player $j$ can see all numbers except her own, $x_j$.
- Every player wants to know if $x_1 = x_2 = \ldots = x_m$ or not.
- Players take turns in **broadcasting** to all other players.
- Recall for $m = 2$, $C(g) = k + 1$.
- What is $C(g)$ for $m \geq 3$?

Consider the following protocol:

- Player 1 broadcasts $A = 1$ if $x_2 = x_3 = \ldots = x_m$, $A = 0$, otherwise.
- If $A = 0$, then all players know that values are not equal.
- If $A = 1$ only player 1 does not know if all values are equal or not; Player 2 broadcasts $B = 1$ if $x_1 = x_3$, $B = 0$, otherwise.

$C(g) = 2$ bits suffice independent of $m$ and $k$!!

Overlap in information can significantly reduce communication.
More on Communication Complexity

- Other definitions of communication complexity:
  - **Average**: Uniform distribution over node values
  - **Randomized**: Nodes use "coin flips" in communication
  - **Non-deterministic**: Non-zero probability of computing error

- Some references:

  A. Yao, "Some complexity questions related to distributive computing," *STOC*, 1979


Information Theory: Lossless Computing

- $m$ node network. Node $j$ observes source $X_j$
- $X_j$ generates i.i.d. process $\{X_{j1}, X_{j2}, \ldots\}$, $X_{ji} \in \mathcal{X}$ a finite set
- Sources may be correlated: $(X_1, X_2, \ldots, X_m) \sim p(x_1, x_2, \ldots, x_m)$
- Node $j$ wishes to find an estimate $\hat{g}_{ji}$ of $g_j(X_{1i}, X_{2i}, \ldots, X_{mi})$ for each sample $i = 1, 2, \ldots$
- **Block coding**: Nodes communicate, perform local computing over blocks of $n$-samples

**Lossless criterion**

$$\hat{g}_j^n = g_j^n \text{ for all } j \in \{1, 2, \ldots, m\} \text{ with high probability (whp)}$$

- What is the minimum sum rate in bits/sample-tuple?
- Problem is in general open, even for 3-nodes
**Lossless Compression**

- Two nodes, one-way communication
- Sender node observes i.i.d. source $X \sim p(x)$
- Receiver node wishes to losslessly estimate $\hat{X}^n$ of $X^n$

**Theorem (Shannon 1948)**

*The minimum lossless compression rate is the entropy of $X$*

$$H(X) := -E(\log p(X)) \text{ bits/sample}$$

**Example (Binary Source)**

$X \sim \text{Bern}(p)$: $H(p) := -p \log p - (1 - p) \log(1 - p) \in [0, 1]$
Random binning proof (Cover 1975)

Ket facts:
- If $x^n$ is generated i.i.d. $\sim p(x)$, then by law of large numbers, its empirical distribution is close to $p(x)$, i.e. it is typical, whp
- There are $\sim 2^{nH(X)}$ typical sequences

Proof outline:
- Randomly assign every sequence $x^n \in X^n$ to one of $2^{nR}$ bins
  Node assignments revealed to both nodes
  
  $$
  \begin{array}{llll}
  \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B}_3 & \ldots \\
  \bullet & \bullet \bullet \bullet & \bullet & \ldots \\
  \text{Typical} & \text{Typical} & \text{Typical} & \ldots \\
  \end{array}
  $$

- Sender: Upon observing $x^n$, send bin index ($nR$ bits)
- Receiver: If $x^n$ is unique typical sequence in bin, set $\hat{x}^n = x^n$; otherwise declare an error

Probability of correct decoding:
- Since $x^n$ is typical whp, bin has $\geq 1$ typical sequences whp
- If $R > H(X)$, bin has 1 typical $x^n$ whp $\Rightarrow \hat{X}^n = X^n$ whp
Lossless Compression with Side Information

- Two nodes, one way communication
- Sender node has i.i.d. source $X$, receiver node has i.i.d. source $Y$
- $(X, Y) \sim p(x, y)$
- Receiver node wishes to estimate $X^n$ losslessly

$H(X)$ sufficient; Can we do better?
- If sender also knows $Y^n$, then minimum rate is conditional entropy

$$H(X|Y) := - \mathbb{E}_{X,Y} (\log p(X|Y)) \leq H(X)$$

Theorem (Slepian-Wolf 1973)

*Minimum rate with side information at receiver only is $H(X|Y)$*
Proof uses random binning

Example (Doubly Symmetric Binary Sources)

\[ Y \sim \text{Bern}(1/2) \text{ and } Z \sim \text{Bern}(p) \text{ be independent, } X = Y \oplus Z \]

Receiver has \( Y \), wishes to losslessly estimate \( X \)

Minimum rate: \[ H(X|Y) = H(Y \oplus Z|Y) = H(Z) = H(p) \]

For example, if \( Z \sim \text{Bern}(0.01) \), \( H(p) \approx 0.08 \)

Versus \( H(X) = H(1/2) = 1 \) if correlation is ignored

Remarks

Shannon theorem holds for error-free compression (e.g., Lempel-Ziv)
Slepian-Wolf does not hold in general for error-free compression
Lossless Computing with Side Information

- Receiver node wishes to compute \( g(X, Y) \)
- Let \( R_g^* \) be minimum rate

\[
R_g^* \leq H(X|Y)
\]
\[
R_g^* \geq H(g(X, Y)|Y)
\]

Upper and lower Bounds
• Bounds sometimes coincide

Example (Mod-2 Sum)

\( Y \sim \text{Bern}(1/2), \ Z \sim \text{Bern}(p) \) independent, \( X = Y \oplus Z \)

Function: \( g(X, Y) = X \oplus Y = Z \)

Lower bound: \( H(X \oplus Y | Y) = H(Z | Y) = H(p) \)

Upper bound: \( H(X | Y) = H(p) \)

\( \Rightarrow R^*_g = H(p) \)

• Bounds do not coincide in general

Example

\( X = (V_1, V_2, \ldots, V_{10}), \ V_j \) i.i.d. Bern(1/2), \( Y \sim \text{Unif}\{1, 2, \ldots, 10\} \)

Function: \( g(X, Y) = V_Y \)

Lower bound: \( H(V_Y | Y) = 1 \) bit

Upper bound: \( H(X | Y) = 10 \) bits

Can show that \( R^*_g = 10 \) bits
Theorem (Orlitsky, Roche 2001)

\[ R^*_g = H_G(X|Y) \]

\( H_G \) is conditional entropy of characteristic graph of \( X, Y \), and \( g \)

Generalizations:

- Two way, two rounds:

- Infinite number of rounds:
  - N. Ma, P. Ishwar, ”Two-terminal distributed source coding with alternating messages for function computation,” ISIT, 2008
**Lossless Computing: $m = 3$**

- 2-sender, 1-receiver network, one-way communication
- Receiver wishes to estimate a function $g(X, Y)$ losslessly
- What is the minimum *sum rate* in bits/sample-pair?

![Diagram](image)

- Problem is in general open
Distributed Lossless Compression

- Let \( g(X, Y) = (X, Y) \)

**Theorem (Slepian-Wolf)**

*The minimum achievable sum rate is the joint entropy*

\[
H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)
\]

- Same as if \( X, Y \) were jointly encoded!
- Achievability uses random binning
- Theorem can be generalized to \( m \) nodes
Distributed Mod-2 Sum Computing

- Let $Y \sim \text{Bern}(1/2)$, $Z \sim \text{Bern}(p)$ independent, $Y = X \oplus Z$
- Receiver wishes to compute $g(X, Y) = X \oplus Y = Z$
- Slepian-Wolf gives sum rate of $H(X, Y) = 1 + H(p)$
- Can we do better?

Theorem (Körner, Marton 1979)

Minimum sum rate is: $2H(p)$

- Minimum sum rate can be $<< 1 + H(p)$
- Achievability uses random linear coding:
  - Randomly generate $k \times n$ binary matrix $A$
  - Senders: Send binary $k$-vectors $AX^n, AY^n$
  - Receiver: Receiver node computes $AX^n \oplus AY^n = AZ^n$
  - If $k > nH(p)$, $AZ^n$ uniquely determines $Z^n$ whp
Information Theory: Lossy Computing

- Node $j$ observes i.i.d. source $X_j$
- Every node wishes to estimate $g(X_{1i}, X_{2i}, \ldots, X_{mi}), i = 1, 2, \ldots$
- Nodes communicate and perform local computing
- As in lossless case, we use block codes

MSE distortion criterion

$$\frac{1}{mn} \sum_{j=1}^{m} \sum_{i=1}^{n} E [(\hat{g}_{ij} - g_i)^2] \leq D$$

- What is the minimum sum rate $R^*(D)$ bits/sample-tuple?
- Problem is in general open (even for 2-nodes)
Lossy Compression

- Two nodes, one-way communication
- Sender node observes i.i.d. Gaussian source $X \sim \mathcal{N}(0, P)$
- Receiver node wishes to estimate $\hat{X}$ to prescribed MSE distortion $D$

What is the minimum required rate $R(D)$?

**Theorem (Shannon 1949)**

Let $d := D/P$ be normalized distortion. The rate distortion function is

$$R(D) = \begin{cases} \frac{1}{2} \log \left( \frac{1}{d} \right) & d \leq 1 \\ 0 & d > 1 \end{cases}$$
Lossy Compression with Side Information

- Two nodes, one-way communication
- \((X, Y)\) Gaussian; 0 mean; average power \(P\); correlation coeff. \(\rho\)
- Receiver wishes to estimate \(X\) to MSE distortion \(D\)

\[
R(D) = \begin{cases} 
\frac{1}{2} \log \left( \frac{1-\rho^2}{d} \right) & d < (1 - \rho^2) \\
0 & \text{otherwise}
\end{cases}
\]

\(d = D/P\)
Lossy Averaging with Side Information

Receiver node now wishes to estimate \((X + Y)/2\) to MSE distortion \(D\)

Proposition

\[
R(D) = \begin{cases} 
\frac{1}{2} \log \left( \frac{1 - \rho^2}{d} \right) & d < (1 - \rho^2) \\
0 & \text{otherwise}
\end{cases}
\]

\(d = 4D/P\)

- Same rate as sending \(X/2\) with MSE distortion \(D\)

For two-way; both nodes wish to estimate average with MSE \(D\)

Proposition (Su, El Gamal 2009)

\[
R(D) = \begin{cases} 
\log \left( \frac{1 - \rho^2}{d} \right) & d < (1 - \rho^2) \\
0 & \text{otherwise}
\end{cases}
\]

- Same as two independent one-way rounds
Distributed Lossy Compression

- \((X, Y)\) jointly Gaussian
- Receiver wishes to estimate \(X, Y\) each with MSE distortion \(D\)
- What is the sum rate distortion function \(R(D)\)?

Problem recently solved (Oohama 97; Wagner, Tavildar, Viswanath 08)
Optimal sum rate known also for \(g = \alpha X + (1 - \alpha) Y, \alpha \in [0, 1]\) (CEO, \(\mu\)-sum (Oohama 05; Prabhakaran, Tse, Ramchandran 04)))
Distributed Consensus: Motivation

Distributed coordination, information aggregation in distributed systems

- Flocking and schooling behaviors in nature
- Spread of rumors, epidemics
- Coordination of mobile autonomous vehicles
- Load balancing in parallel computers
- Finding file sizes, free memory size in peer-to-peer networks
- Information aggregation in sensor networks

In many of these scenarios, communication is asynchronous, subject to link and node failures and topology changes.
Distributed Averaging

- $m$-node network. Node $j$ observes real-valued scalar $x_j$
- Every node wishes to estimate the average $(1/m) \sum_{j=1}^{m} x_j$
- Communication performed in rounds, for example,
  - 2 nodes selected in each round
  - They exchange information and perform local computing

- **Distributed Protocol**: Communication and local computing do not depend on node identity
  - **Synchronous (deterministic)**: Node-pair selection predetermined
  - **Asynchronous (Gossip)**: Node-pairs selected at random

How many rounds are needed to estimate the average (time-to-consensus)?
Recent Related Work


Example: Centralized Protocol

- Node 1 acts as cluster-head
- **Rounds 1–4**: Cluster-head receives values from other nodes
- The cluster head computes the average
- **Rounds 5–8**: Cluster-head sends average to other nodes
- The number of rounds for \( m \)-node network: \( 2m - 2 \)
- Computation is perfect
Example: Gossip Protocol

- At each round, a node-pair \((j, k)\) is randomly selected.
- Nodes exchange values and average them.
- Computation is not perfect: \(\text{MSE} = 0.00875\).
- In general, cannot achieve perfect computation in finite number of rounds.
- Estimates converge to true average.
Issues

- Communicating/computing with infinite precision not realistic
- Number of rounds not good measure of communication cost

Quantized consensus:
- M. Yildiz, A. Scaglione, ”Coding with side information for rate constrained consensus,” *IEEE Trans. on Sig. Proc.*, 2008

- We recently studied lossy computing formulation:
Distributed Lossy Averaging (Su, El Gamal 2009)

- Network with $m$ nodes
- Node $j$ observes Gaussian i.i.d. source $X_j \sim \mathcal{N}(0, 1)$
- Assume sources are independent
- Each node wishes to estimate the average $g^n := (1/m) \sum_{j=1}^{m} X_j^n$
- Rounds of two-way, node-pair communication; block codes used

Network rate distortion function

$R^*(D)$ is minimum sum rate such that

$$\frac{1}{mn} \sum_{j=1}^{m} \sum_{i=1}^{n} \mathbb{E} \left[ (\hat{g}_{ij} - g_i)^2 \right] \leq D$$

- $R^*(D)$ known only for $m = 2$
Cutset Lower Bound on $R^*(D)$

Super-node

\[ P = \frac{m-1}{m^2} \]

Theorem (Cutset Lower Bound (Su, El Gamal 2009))

\[ R^*(D) \geq \frac{m}{2} \log \left( \frac{m-1}{m^2 D} \right) \quad \text{for } D < \frac{(m-1)}{m^2} \]

- Bound is tight for $m = 2$
- Can achieve within factor of 2 using centralized protocol for large $m$
Gossip Protocol

- Two nodes are selected at random in each round
- Use weighted-average-and-compress scheme
- Before communication, node $j$ sets $\hat{g}_j^n(0) = X_j^n$
- At round $t+1$: Suppose nodes $j$, $k$ selected:
  - Nodes exchange lossy estimates of their states $\hat{g}_j^n(t)$, $\hat{g}_k^n(t)$ at normalized distortion $d$
  - Nodes update their estimates:
    \[
    \begin{align*}
    \hat{g}_j^n(t + 1) &= \frac{1}{2} \hat{g}_j^n(t) + \frac{1}{2(1 - d)} \hat{g}_k^n(t), \\
    \hat{g}_k^n(t + 1) &= \frac{1}{2} \hat{g}_k^n(t) + \frac{1}{2(1 - d)} \hat{g}_j^n(t)
    \end{align*}
    \]
- Final estimate for node $j$ is $\hat{g}_j^n(T)$
- Rate at each round is $2 \times (1/2) \log(1/d)$
- Update equations reduce to gossip when $d = 0$, rate $\rightarrow \infty$
Gossip Protocol

- Let $E(R(D))$ be expected sum rate over node-pair selection
- $R^*(D) \leq E(R(D))$
- Can show that $E(R(D))$ larger than cutset bound by roughly a factor of $\log m$
- Cutset bound achievable within factor to 2 using centralized protocol

**Price of gossiping is roughly $\log m$**

- Protocol does not exploit build-up in correlation to reduce rate
Effect of Using Correlation ($m = 50$)
## Summary

- Communication cost of distributed computing studied in several fields

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<th>Coding/Computing</th>
<th>Estimation Criterion</th>
<th>Communication Cost</th>
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- Many cool results, but much remains to be done
Thank You