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An Achievable Rate Region for the 3-User-Pair Deterministic Interference Channel

Invited Paper

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***Abstract*—This paper presents a new coding scheme and a corresponding achievable rate region for a class of deterministic interference channels with three sender–receiver pairs (3-DIC). The codebook structure uses rate splitting, Marton coding, and superposition coding. The receivers use interference decoding to exploit the structure of the combined interference signal without uniquely decoding any of the interfering messages partly or fully. The corresponding rate region is shown to contain all previously known achievable rate regions for the deterministic interference channel. The scheme can be extended to general discrete memoryless interference channels and provides a natural generalization of the Han–Kobayashi scheme for the two-user-pair case.**

I. Introduction

In defiance of years of investigation, the capacity region of the interference channel remains unknown in general. This is the case even for the two sender–receiver pair interference channel. The best known inner bound for the memoryless 2-pair interference channel is achieved by the Han–Kobayashi coding scheme [1, 2], which divides each message into a common part (which is decoded at both receivers) and a private part (which is decoded only at the desired receiver). Codebooks are then constructed by superposition coding. A natural question to ask is how to generalize the Han–Kobayashi scheme to interference channels with more than two user pairs. For the Gaussian case, it was shown in [3] that a straightforward extension using a partial message for each subset of receivers and superposition coding does not work well in general.

In this paper, we investigate this question for the three-pair deterministic interference channel (3-DIC) depicted in Figure 1, which was first introduced in [4, 5]. The channel consists of three sender–receiver alphabet pairs $(\mathcal{X}_l, \mathcal{Y}_l)$, loss functions g_{lk}, interference combining functions h_l, and receiver functions f_l for $l, k \in \{1, 2, 3\}$. The channel is memoryless, and its outputs are

$$
\begin{aligned}
Y_l &= f_l(X_{ll}, S_l), \quad \text{where} \\
X_{lk} &= g_{lk}(X_l), \\
S_1 &= h_1(X_{21}, X_{31}), \\
S_2 &= h_2(X_{32}, X_{12}), \\
S_3 &= h_3(X_{13}, X_{23}).
\end{aligned}
$$

We assume that h_l and f_l are injective in each argument, that is, they become one-to-one when either one of their arguments is fixed. For example, for $Y_1 = f_1(X_{11}, S_1)$, this assumption is equivalent to $H(X_{11}) = H(Y_1 \mid S_1)$ and $H(S_1) = H(Y_1 \mid X_{11})$ for every probability mass function (pmf) $p(x_{11}, s_1)$. Each sender $l \in \{1, 2, 3\}$ wishes to convey an independent message M_l at data rate R_l to its corresponding receiver. We define a $(2^{nR_1}, 2^{nR_2}, 2^{nR_3}, n)$ code, probability of error, achievability of a given rate triple (R_1, R_2, R_3), and the capacity region in the standard way (see [6]).

This channel is a natural choice to consider for several reasons. First, it allows us to explore the effect of interference without noise. Second, it generalizes the two-pair deterministic interference channel in [7] for which the capacity region is known and is achieved by the Han–Kobayashi scheme. Third, certain deterministic interference channels have been shown to be good approximations of the Gaussian interference channel in high SNR [8], which is of great interest in wireless communication. Finally, the presentation and analysis of our scheme for the deterministic case is somewhat simpler than for the general interference channel.

In the following section we present a new coding scheme for the 3-DIC and a corresponding inner bound to the capacity region. This coding scheme generalizes the Han–Kobayashi scheme, and performs strictly better than previously known inner bounds on the 3-DIC capacity region. The key idea is to combine the receiver-centric insight obtained from interference

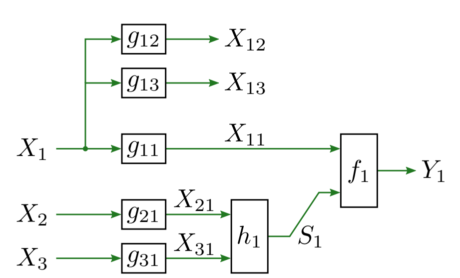

Fig. 1. Block diagram of the 3-DIC for the first receiver.

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decoding [5] with the transmitter-centric viewpoint that led to the setting of communication with disturbance constraints [9]. We borrow the codebook construction via Marton coding and superposition coding from the latter, and apply saturation arguments at the receiver similar to the former, which permits us to take advantage of the structure of the combined interfering signal without decoding any of the interfering messages partly or fully. An example is presented in Subsection III. The proofs, some of which we can only sketch due to space limitation, are relegated to Section IV.

II. Inner bound to the 3-DIC capacity region

In this section, we state our main result. Fix a joint pmf over $(Q, U_1, X_1, U_2, X_2, U_3, X_3)$ of the form

$$p = p(q)p(u_1, x_1|q)p(u_2, x_2|q)p(u_3, x_3|q).$$

Here, Q and U_l, for $l \in [3]$, are auxiliary random variables of arbitrary cardinality.[1] While Q is a time-sharing random variable that is common between all three transmitters, U_l is associated with the lth transmitter only. Define the rate region $\mathscr{R}_1(p) \subset \mathbb{R}_+^{18}$ to consist of the rate tuples

$$\begin{aligned}(&R_{10}, R_{11}, R_{12}, R_{13}, \tilde{R}_{12}, \tilde{R}_{13},\\ &R_{20}, R_{22}, R_{23}, R_{21}, \tilde{R}_{23}, \tilde{R}_{21},\\ &R_{30}, R_{33}, R_{31}, R_{32}, \tilde{R}_{31}, \tilde{R}_{32})\end{aligned} \tag{1}$$

such that

$$\tilde{R}_{12} - R_{12} + \tilde{R}_{13} - R_{13} \geq I(X_{12}; X_{13} \,|\, U_1, Q), \tag{2}$$

$$\tilde{R}_{12} - R_{12} + (\tilde{R}_{13} - R_{13})/2 \leq I(X_{12}; X_{13} \,|\, U_1, Q), \tag{3}$$

$$(\tilde{R}_{12} - R_{12})/2 + \tilde{R}_{13} - R_{13} \leq I(X_{12}; X_{13} \,|\, U_1, Q), \tag{4}$$

$$\tilde{R}_{12} \geq R_{12}, \tag{5}$$

$$\tilde{R}_{13} \geq R_{13}, \tag{6}$$

and for all $i \in [5]$,

$$r_{1i} \leq H(X_{11} \,|\, c_{1i}, Q) + t_{1i}, \tag{7}$$

$$r_{1i} + \tilde{R}_{21} \leq H(Y_1 \,|\, c_{1i}, U_2, X_{31}, Q) + t_{1i}, \tag{8}$$

$$r_{1i} + \tilde{R}_{31} \leq H(Y_1 \,|\, c_{1i}, X_{21}, U_3, Q) + t_{1i}, \tag{9}$$

$$\begin{aligned}r_{1i} + \min\{&R_{20} + \tilde{R}_{21}, H(X_{21} \,|\, Q)\}\\ &\leq H(Y_1 \,|\, c_{1i}, X_{31}, Q) + t_{1i},\end{aligned} \tag{10}$$

$$\begin{aligned}r_{1i} + \min\{&R_{30} + \tilde{R}_{31}, H(X_{31} \,|\, Q)\}\\ &\leq H(Y_1 \,|\, c_{1i}, X_{21}, Q) + t_{1i},\end{aligned} \tag{11}$$

$$\begin{aligned}r_{1i} + \min\{&\tilde{R}_{21} + \tilde{R}_{31}, H(S_1 \,|\, U_2, U_3, Q)\}\\ &\leq H(Y_1 \,|\, c_{1i}, U_2, U_3, Q) + t_{1i},\end{aligned} \tag{12}$$

$$\begin{aligned}r_{1i} + \min\{&R_{20} + \tilde{R}_{21} + \tilde{R}_{31},\\ &H(X_{21} \,|\, Q) + \tilde{R}_{31}, H(S_1 \,|\, U_3, Q)\}\\ &\leq H(Y_1 \,|\, c_{1i}, U_3, Q) + t_{1i},\end{aligned} \tag{13}$$

$$\begin{aligned}r_{1i} + \min\{&\tilde{R}_{21} + R_{30} + \tilde{R}_{31},\\ &\tilde{R}_{21} + H(X_{31} \,|\, Q), H(S_1 \,|\, U_2, Q)\}\\ &\leq H(Y_1 \,|\, c_{1i}, U_2, Q) + t_{1i},\end{aligned} \tag{14}$$

$$\begin{aligned}r_{1i} + \min\{&R_{20} + \tilde{R}_{21} + R_{30} + \tilde{R}_{31}, R_{20} + \tilde{R}_{21} +\\ &H(X_{31} \,|\, Q), H(X_{21} \,|\, Q) + R_{30} + \tilde{R}_{31}, H(S_1 \,|\, Q)\}\\ &\leq H(Y_1 \,|\, c_{1i}, Q) + t_{1i}.\end{aligned} \tag{15}$$

In the latter set of conditions, lower-case symbols are placeholders for the terms specified in Table 1. The term r_{1i} represents rates, the term c_{1i} stands for sets of random variables on which certain entropy terms are conditioned, and t_{1i} is an additive penalty term. For example, with $i = 3$, condition (13) corresponds to the inequality

$$\begin{aligned}\tilde{R}_{13} + R_{11} + \min\big\{&R_{20} + \tilde{R}_{21} + \tilde{R}_{31},\\ &H(X_{21} \,|\, Q) + \tilde{R}_{31},\\ &H(S_1 \,|\, U_3, Q)\big\}\\ \leq H(Y_1 \,|\, U_1, X_{12}, U_3, Q) &+ I(X_{12}; X_{13} \,|\, U_1, Q).\end{aligned} \tag{16}$$

Similarly, define the regions $\mathscr{R}_2(p)$ and $\mathscr{R}_3(p)$ by making the subscript replacements $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $1 \mapsto 3 \mapsto 2 \mapsto 1$ in the definition of $\mathscr{R}_1(p)$, respectively.

Define an operator FM that maps a convex 18-dimensional set of rate vectors of the form (1) to a 3-dimensional rate region by substituting $R_{l0} = R_l - R_{l1} - R_{l2} - R_{l3}$, for $l \in [3]$, and subsequently projecting on the coordinates (R_1, R_2, R_3). The operator FM can be implemented by Fourier–Motzkin elimination.

Let $\overline{\mathscr{S}}$ be the convex hull of the set $\mathscr{S}$. We are now ready to state the main result.

Theorem 1. The region

$$\mathscr{R}_{\text{IB}} = \overline{\bigcup_p \text{FM}\left\{\overline{\mathscr{R}_1(p) \cap \mathscr{R}_2(p) \cap \mathscr{R}_3(p)}\right\}},$$

where p is of the form $p(q)p(u_1, x_1|q)p(u_2, x_2|q)p(u_3, x_3|q)$, is an inner bound to the capacity region of the 3-DIC.

Remark 1 (Saturation). The min terms on the left hand side of conditions (7) to (15) correspond to different modes of signal saturation, as in interference decoding [5]. There are numerous modes of saturation here, since the transmitters employ a more sophisticated scheme than single-user random codes.

Remark 2 (Convexity). The regions $\mathscr{R}_1(p)$, $\mathscr{R}_2(p)$, and $\mathscr{R}_3(p)$ ensure decodability at the first, second, and third receiver, respectively. As in [5], they are generally non-convex. The regions can alternatively be written as finite unions of convex components by expanding the cases in which the min terms take on each of their arguments.

i	r_{1i}	c_{1i}	t_{1i}
1	R_{11}	$\{U_1, X_{12}, X_{13}\}$	0
2	$\tilde{R}_{12} + R_{11}$	$\{U_1, X_{13}\}$	I_1
3	$\tilde{R}_{13} + R_{11}$	$\{U_1, X_{12}\}$	I_1
4	$\tilde{R}_{12} + \tilde{R}_{13} + R_{11}$	$\{U_1\}$	I_1
5	$R_{10} + \tilde{R}_{12} + \tilde{R}_{13} + R_{11}$	$\emptyset$	I_1

Table 1. Shorthand notation for terms related to transmitter 1. In the fourth column, I_1 stands for $I(X_{12}; X_{13} \,|\, U_1, Q)$.

[1]The notation [3] is shorthand for the set $\{1, 2, 3\}$.

The intersection $\mathscr{R}_1(p) \cap \mathscr{R}_2(p) \cap \mathscr{R}_3(p)$ is also generally non-convex. By virtue of time-sharing, we are allowed to convexify. (This convex hull operation is not achieved by the coded time sharing mechanism of Q.) The explicit convex hull operation also ensures that the argument of FM is in fact a convex set.

Remark 3 (Fourier–Motzkin elimination). In contrast to other settings that use rate splitting, the Fourier–Motzkin elimination denoted by FM cannot be carried out symbolically. Formally, this is due to the convex hull operation in its argument. However, this does not hinder numerical evaluation of the region, since for each fixed p, the set $\overline{\mathscr{R}_1(p) \cap \mathscr{R}_2(p) \cap \mathscr{R}_3(p)}$ as represented by its extreme points can be computed explicitly, and FM can be evaluated by numerical Fourier–Motzkin elimination.

Remark 4 (Optimality). It is not known whether the inner bound is tight in general. It strictly includes the interference decoding inner bound in [5], and thereby, the bound obtained by single-user random codes and treating interference as noise. This follows by setting $U_l = X_l$ and $R_l = R_{l0}$ for all $l \in [3]$.

Furthermore, the two-pair projections of the inner bound are optimal, i.e., if one of the three rates, say R_3, is set to zero, the two-dimensional region that the inner bound achieves for (R_1, R_2) is in fact the capacity region of the interference channel that consists of the first and second user pair. This follows by setting $U_l = \emptyset$ for all $l \in [3]$, letting $\tilde{R}_{12} = R_{12}$, $\tilde{R}_{13} = R_{13} + I(X_{12}; X_{13} \,|\, Q)$, $\tilde{R}_{21} = R_{21}$, $\tilde{R}_{23} = R_{23} + I(X_{21}; X_{23} \,|\, Q)$, and replacing all min terms in (7) to (15) with their first argument. In fact, the codebook structure that underlies the inner bound contains superposition codebooks as a special case. Hence the proposed coding scheme subsumes the Han–Kobayashi coding scheme and generalizes it naturally to more than two user pairs.

Alternative characterization of the inner bound

We present a modified achievable region. While more difficult to compute than the region of Theorem 1, this formulation allows deeper insight into the structure of the decodability conditions (see Remark 6).

Define a new region $\mathscr{R}'_1(p)$ similar to $\mathscr{R}_1(p)$ above, but replacing conditions (7) to (15) by

$$r_{1i} + \min\{r_{21j} + r_{31k}, H(S_1 \,|\, c_{21j}, c_{31k}, Q)\} \le H(Y_1 \,|\, c_{1i}, c_{21j}, c_{31k}, Q) + t_{1i}, \quad \text{for all } i \in [5], j \in [3], k \in [3]. \tag{17}$$

The lower-case symbols indexed by i, j, and k are placeholders for the terms specified in Tables 1, 2, and 3, respectively. For example, the case where $i = 3$, $j = 3$, and $k = 2$ corresponds to the inequality

$$\tilde{R}_{13} + R_{11} + \min\{\min\{R_{20} + \tilde{R}_{21}, R_{20} + H(X_{21} \,|\, U_2, Q), H(X_{21} \,|\, Q)\} + \min\{\tilde{R}_{31}, H(X_{31} \,|\, U_3, Q)\},\ H(S_1 \,|\, U_3, Q)\} \le H(Y_1 \,|\, U_1, X_{12}, U_3, Q) + I(X_{12}; X_{13} \,|\, U_1, Q). \tag{18}$$

Similarly, define the regions $\mathscr{R}'_2(p)$ and $\mathscr{R}'_3(p)$ by making the subscript replacements $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $1 \mapsto 3 \mapsto 2 \mapsto 1$ in the definition of $\mathscr{R}'_1(p)$, respectively.

j	r_{21j}	c_{21j}
1	0	$\{X_{21}\}$
2	$\min\{\tilde{R}_{21}, H(X_{21} \,|\, U_2, Q)\}$	$\{U_2\}$
3	$\min\{R_{20} + \tilde{R}_{21}, R_{20} + H(X_{21} \,|\, U_2, Q), H(X_{21} \,|\, Q)\}$	$\emptyset$

Table 2. Shorthand notation for terms related to transmitter 2.

k	r_{31k}	c_{31k}
1	0	$\{X_{31}\}$
2	$\min\{\tilde{R}_{31}, H(X_{31} \,|\, U_3, Q)\}$	$\{U_3\}$
3	$\min\{R_{30} + \tilde{R}_{31}, R_{30} + H(X_{31} \,|\, U_3, Q), H(X_{31} \,|\, Q)\}$	$\emptyset$

Table 3. Shorthand notation for terms related to transmitter 3.

Corollary 1. The region

$$\mathscr{R}'_{\text{IB}} = \overline{\bigcup_p \text{FM}\left\{\overline{\mathscr{R}'_1(p) \cap \mathscr{R}'_2(p) \cap \mathscr{R}'_3(p)}\right\}},$$

where p is of the form $p(q)p(u_1, x_1|q)p(u_2, x_2|q)p(u_3, x_3|q)$, is an inner bound to the capacity region of the 3-DIC.

Remark 5. The regions $\mathscr{R}'_{\text{IB}}$ and $\mathscr{R}_{\text{IB}}$ of Corollary 1 and Theorem 1 are equal. This is proved in Subsection IV-D.

Remark 6 ($\mathscr{R}'_{\text{IB}}$ is logically simpler than $\mathscr{R}_{\text{IB}}$). Inequality (17) exposes a *product structure* with individual "factors" related to the first, second, and third transmitter as specified in Tables 1, 2, and 3, respectively. This structure reflects the fact that the transmitted messages are independent and there is no cooperation between the transmitting nodes.

Remark 7 ($\mathscr{R}_{\text{IB}}$ is computationally simpler than $\mathscr{R}'_{\text{IB}}$). The sets $\mathscr{R}_1(p)$ and $\mathscr{R}'_1(p)$ are both defined by 50 inequality conditions. There is a natural one-to-one correspondence between inequalities for $\mathscr{R}_1(p)$ and $\mathscr{R}'_1(p)$ in which conditions (7) through (15) for some index i correspond to inequality (17) for the same i and all $j, k \in [3]$. However, the individual conditions are much simpler for $\mathscr{R}_1(p)$ than for $\mathscr{R}'_1(p)$. Consider, for example, the corresponding conditions (16) and (18). Expanding the nested min terms in (18) leads to

$$\begin{aligned}\tilde{R}_{13} + R_{11} + \min\{&R_{20} + \tilde{R}_{21} + \tilde{R}_{31},\\ &R_{20} + \tilde{R}_{21} + H(X_{31} \,|\, U_3, Q),\\ &R_{20} + H(X_{21} \,|\, U_2, Q) + \tilde{R}_{31},\\ &R_{20} + H(X_{21} \,|\, U_2, Q) + H(X_{31} \,|\, U_3, Q),\\ &H(X_{21} \,|\, Q) + \tilde{R}_{31},\\ &H(S_1 \,|\, U_3, Q)\}\\ \le H(Y_1 \,|\, U_1, X_{12}, U_3, Q) &+ I(X_{12}; X_{13} \,|\, U_1, Q).\end{aligned} \tag{19}$$

The difference between the expression in (16) and the one in (19) is that the former has fewer arguments in the min term than the latter. The non-convex set $\mathscr{R}_1(p)$ therefore consists of fewer convex components than $\mathscr{R}'_1(p)$, which reduces the computational effort required to evaluate the region.

III. Example

Consider the cyclically symmetric 3-DIC depicted in Figure 2. This is the same example as studied in [5]. Figure 3(a) depicts the achievable rate region with interference decoding, while Figure 3(b) contains a numerical approximation of the inner bound in Theorem 1. For simplicity, the region is evaluated with $U_l = \emptyset$ for $l \in [3]$.

The optimal trade-off between R_1 and R_2 when $R_3 = 0$ is achieved in Figure 3(b), as per Remark 4. The same is not true for the interference decoding inner bound in Figure 3(a). Figure 3(c) depicts the intersection of the three-dimensional regions with the plane defined by the R_2 axis and the 45°-line between the R_1 and R_3 axes. This plane is also depicted in Figure 3(b). The intersection highlights the improvement of Theorem 1 over the interference decoding inner bound.

IV. Proofs

We divide the proof of Theorem 1 and Corollary 1 into four parts. Both results are based on the same coding scheme, which is described in Subsection IV-A. In Subsection IV-B, we give a lemma and two corollaries that bound the probability of a union of events. These are then applied during the error probability analysis that is sketched in Subsection IV-C and establishes Corollary 1. Finally, in Subsection IV-D, we show that the regions in Theorem 1 and Corollary 1 are in fact equal, which concludes the proof of the theorem.

For the sake of simplified notation, we omit the auxiliary random variable Q throughout this section. To obtain the proof with Q, the codebook generation procedure as described below must be augmented by generating a coded time sharing sequence q^n i.i.d. from $p(q)$, and conditioning all subsequent analysis steps on it in the usual way [6].

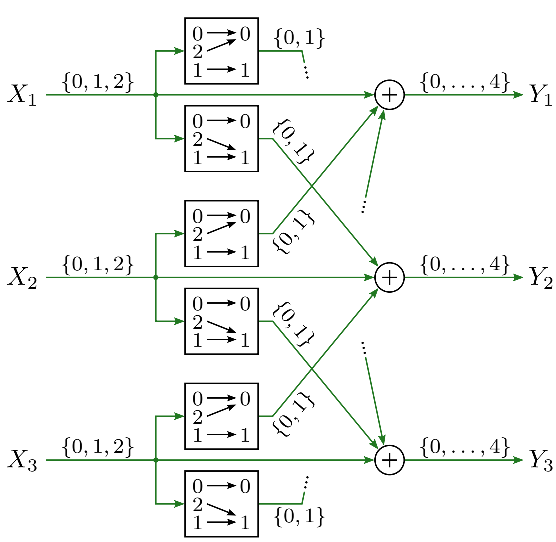
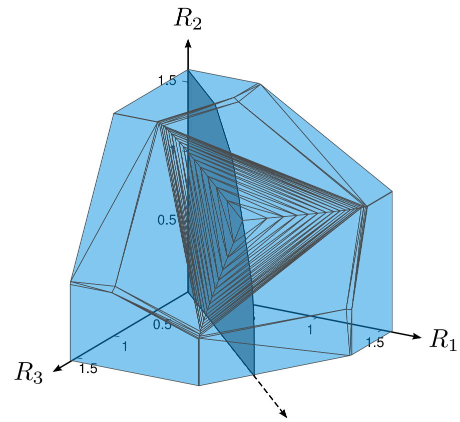
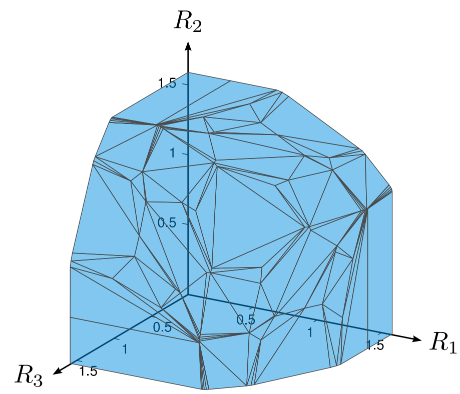


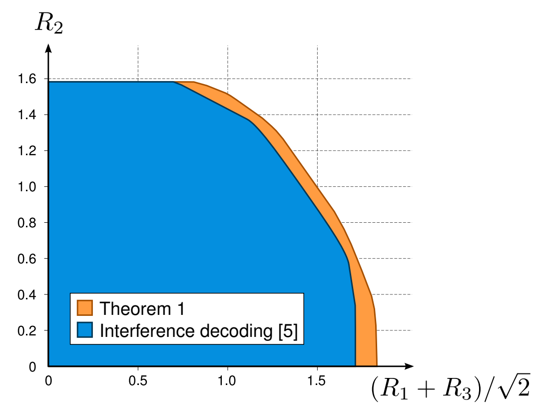
Fig. 2. Additive 3-DIC example.



(a) Interference decoding inner bound in [5]



(b) Inner bound of Theorem 1



(c) Intersection with 45°-plane

Fig. 3. Inner bounds for the additive 3-DIC example.

A. Codebook generation

Fix a pmf $p(u_1, x_1)p(u_2, x_2)p(u_3, x_3)$. We begin by describing the generation procedure for the first transmitter. The codebook is constructed as in the deterministic case of communication with two disturbance constraints [9], using Marton coding and superposition coding. The crosslink outputs X_{12} and X_{13} take the place of the side receivers that are not interested in any part of the message.

Split the rate as $R_1 = R_{10} + R_{11} + R_{12} + R_{13}$. Define the auxiliary rates $\tilde{R}_{12} \geq R_{12}$ and $\tilde{R}_{13} \geq R_{13}$, let $\varepsilon' > 0$, and define the set partitions

$$[2^{n\tilde{R}_{12}}] = \mathcal{L}_{12}(1) \cup \cdots \cup \mathcal{L}_{12}(2^{nR_{12}}),$$
$$[2^{n\tilde{R}_{13}}] = \mathcal{L}_{13}(1) \cup \cdots \cup \mathcal{L}_{13}(2^{nR_{13}}),$$

where $\mathcal{L}_{12}(\cdot)$ and $\mathcal{L}_{13}(\cdot)$ are indexed sets of size $2^{n(\tilde{R}_{12}-R_{12})}$ and $2^{n(\tilde{R}_{13}-R_{13})}$, respectively.

1) For each $m_{10} \in [2^{nR_{10}}]$, generate $u_1^n(m_{10})$ according to $\prod_{i=1}^n p(u_{1i})$.
2) For each $l_{12} \in [2^{n\tilde{R}_{12}}]$, generate $x_{12}^n(m_{10}, l_{12})$ according to $\prod_{i=1}^n p(x_{12i} \,|\, u_{1i}(m_{10}))$. Likewise, for each $l_{13} \in [2^{n\tilde{R}_{13}}]$, generate $x_{13}^n(m_{10}, l_{13})$ according to $\prod_{i=1}^n p(x_{13i} \,|\, u_{1i}(m_{10}))$.
3) For each (m_{10}, m_{12}, m_{13}), let $\mathcal{S}(m_{10}, m_{12}, m_{13})$ be the set of all pairs (l_{12}, l_{13}) from the product set $\mathcal{L}_{12}(m_{12}) \times \mathcal{L}_{13}(m_{13})$ such that $(x_{12}^n(m_{10}, l_{12}), x_{13}^n(m_{10}, l_{13})) \in \mathcal{T}_{\varepsilon'}^{(n)}(X_{12}, Z_{13} \,|\, u_1^n(m_{10}))$.
4) For each (m_{10}, l_{12}, l_{13}) and $m_{11} \in [2^{nR_{11}}]$, generate $x_1^n(m_{10}, l_{12}, l_{13}, m_{11})$ according to
$$\prod_{i=1}^n p(x_{1i} \,|\, u_{1i}(m_{10}), x_{12i}(l_{12}), x_{13i}(l_{13}))$$
if $(l_{12}, l_{13}) \in \mathcal{S}(m_{10}, m_{12}, m_{13})$. Otherwise, we draw from $\mathrm{Unif}(\mathcal{X}^n)$.
5) Choose $(l_{12}^{(m_{10},m_{12},m_{13})}, l_{13}^{(m_{10},m_{12},m_{13})})$ uniformly from $\mathcal{S}(m_{10}, m_{12}, m_{13})$. If $\mathcal{S}(m_{10}, m_{12}, m_{13})$ is empty, choose $(1, 1)$.

Note the notational difference in the way the rates R_{1i} are indexed here as opposed to [9]. Here, the steps in codebook generation follow the scheme $R_{10} \to (R_{12}, R_{13}) \to R_{11}$. The first index of the rate variables is always 1 and represents the first transmitter. The second index uses the intuition that R_{10} is "common" in the sense that it affects both side receivers, and R_{11} is "private" and does not appear at either side receiver. (In [9], there is only one transmitter and thus no first index. The second index follows the notational convention $R_0 \to (R_1, R_2) \to R_3$.)

Codebooks for the second and third transmitter are generated analogously by applying the subscript replacements $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $1 \mapsto 3 \mapsto 2 \mapsto 1$ in each step of the procedure.

Encoding. To send message $m_1 = (m_{10}, m_{12}, m_{13}, m_{11})$, transmit $x_1^n(m_{10}, l_{12}^{(m_{10},m_{12},m_{13})}, l_{13}^{(m_{10},m_{12},m_{13})}, m_{11})$.

Decoding. The first receiver observes y_1^n. Define the tuple

$$\begin{aligned} &T(m_{10}, m_{12}, m_{13}, m_{11}, m_{20}, l_{21}, m_{30}, l_{31}) \\ &= \Big(u_1^n(m_{10}), x_{12}^n(m_{10}, l_{12}^{(m_{10},m_{12},m_{13})}), \\ &\quad x_{13}^n(m_{10}, l_{13}^{(m_{10},m_{12},m_{13})}), \\ &\quad x_1^n(m_{10}, l_{12}^{(m_{10},m_{12},m_{13})}, l_{13}^{(m_{10},m_{12},m_{13})}, m_{11}), \\ &\quad u_2^n(m_{20}), x_{21}^n(m_{20}, l_{21}), u_3^n(m_{30}), x_{31}^n(m_{30}, l_{31}), \\ &\quad s_1^n(m_{20}, l_{21}, m_{30}, l_{31}), y_1^n\Big). \end{aligned}$$

Let $\varepsilon > \varepsilon'$. Declare that $\hat{m}_1 = (\hat{m}_{10}, \hat{m}_{12}, \hat{m}_{13}, \hat{m}_{11})$ has been sent if it is the unique message such that

$$\begin{aligned} &T(\hat{m}_{10}, \hat{m}_{12}, \hat{m}_{13}, \hat{m}_{11}, \mathring{m}_{20}, \mathring{l}_{21}, \mathring{m}_{30}, \mathring{l}_{31}) \\ &\qquad \in \mathcal{T}_{\varepsilon}^{(n)}(U_1, X_{12}, X_{13}, X_1, U_2, X_{21}, U_3, X_{31}, S_1, Y_1) \end{aligned}$$

for some $\mathring{m}_{20}, \mathring{l}_{21}, \mathring{m}_{30}, \mathring{l}_{31}$.

B. Probability decomposition by index and by value

In this subsection, we state a lemma and two corollaries that are useful in the error probability analysis in Subsection IV-C. They bound the probability of a union of events, such as the probability that *any* one of the possible incorrect message combinations appears to be correct at a receiver in the 3-DIC. For brevity's sake, the proofs are omitted.

Lemma 1. Let $A_1, \ldots, A_n$ be identically distributed random variables from an alphabet $\mathcal{A}$, and let D be a random variable from alphabet $\mathcal{D}$. Let $\mathcal{Q} \subset \mathcal{A} \times \mathcal{D}$ be a set of "qualified" pairs. Then

$$\mathrm{P}\left(\cup_{i=1}^n \{(A_i, D) \in \mathcal{Q}\}\right) \leq \sum_{i=1}^n \mathrm{P}((A_i, D) \in \mathcal{Q}), \qquad (20)$$

$$\mathrm{P}\left(\cup_{i=1}^n \{(A_i, D) \in \mathcal{Q}\}\right) \leq \sum_{a \in \mathcal{A}} \mathrm{P}((a, D) \in \mathcal{Q}). \qquad (21)$$

Remark 8. Inequality (20) is the well-known union bound; it decomposes the probability *by index*. The second inequality (21) decomposes the probability *by value*.

Remark 9. Note that the random variable D is crucial in inequality (21). Without D, i.e., $D = \emptyset$, the terms in the sum are essentially indicator functions, and the right hand side of the bound generally becomes larger than one and thus useless. For the bound to be useful, the randomness of D must act as dithering that equalizes the probability $\mathrm{P}((a, D) \in \mathcal{Q})$ over a.

For the following two corollaries, in addition to the assumptions in Lemma 1, let $\mathcal{Q}_A$ be the subset of values from $\mathcal{A}$ that can qualify at all[2], and let P_D be given such that $\mathrm{P}((a, D) \in \mathcal{Q}) \leq P_D$ for all a.

Corollary 2.

$$\mathrm{P}\left(\cup_{i=1}^n \{(A_i, D) \in \mathcal{Q}\}\right) \leq n\,\mathrm{P}(A_i \in \mathcal{Q}_A) \cdot P_D.$$

Corollary 3.

$$\mathrm{P}\left(\cup_{i=1}^n \{(A_i, D) \in \mathcal{Q}\}\right) \leq |\mathcal{Q}_A| \cdot P_D.$$

Remark 10. The factor $n\,\mathrm{P}(A_i \in \mathcal{Q}_A)$ in Corollary 2 is the expected number of random variables A_i that qualify by themselves (for some d). It thus relates to counting random variables, which matches our interpretation of enumerating random variable *indices*. On the other hand, the factor $|\mathcal{Q}_A|$ in Corollary 3 counts a set of *values* of random variables.

[2] $\mathcal{Q}_A = \{a \in \mathcal{A} \,|\, (a, d) \in \mathcal{Q} \text{ for some } d\}$

C. Sketch of error probability analysis for Corollary 1

Without loss of generality, assume that $m_{l0} = m_{l1} = m_{l2} = m_{l3} = 1$ is transmitted from users $l \in [3]$. To analyze the probability of error at the first receiver, define the following events.

$$\begin{aligned}
\mathcal{E}_{\mathrm{e}1}: &\quad \mathcal{S}(1,1,1) \text{ is empty},\\
\mathcal{E}_{\mathrm{e}2}: &\quad \mathcal{S}(1,1,1) \text{ contains two distinct pairs with equal first or second component},\\
\mathcal{E}_0: &\quad \{T(1,1,1,1,m_{20},l_{21},m_{30},l_{31}) \notin \mathcal{T}_\varepsilon^{(n)} \text{ for all } m_{20},l_{21},m_{30},l_{31}\},\\
\mathcal{E}_{ijk}: &\quad \{T(m_{10},m_{12},m_{13},m_{11},m_{20},l_{21},m_{30},l_{31}) \in \mathcal{T}_\varepsilon^{(n)} \text{ for some } (m_{10},m_{12},m_{13},m_{11}) \in \mathcal{M}_{1i},\\
&\quad (m_{20},l_{21}) \in \mathcal{M}_{2j}, (m_{30},l_{31}) \in \mathcal{M}_{3k}\}, \quad \text{for } i \in [5], j \in [3], k \in [3],
\end{aligned}$$

where the message subsets $\mathcal{M}_{1i}$, $\mathcal{M}_{2j}$, and $\mathcal{M}_{3k}$ are specified in Tables 4, 5, and 6. With "encoding" and "decoding" error events

$$\mathcal{E}_{\mathrm{e}} = \mathcal{E}_{\mathrm{e}1} \cup \mathcal{E}_{\mathrm{e}2},$$
$$\mathcal{E}_{\mathrm{d}} = \mathcal{E}_0 \cup \bigcup_{i,j,k} \mathcal{E}_{ijk},$$

the probability of error is upper bounded by

$$\mathrm{P}(\mathcal{E}) \le \mathrm{P}(\mathcal{E}_{\mathrm{e}}) + \mathrm{P}(\mathcal{E}_{\mathrm{d}} \,|\, \mathcal{E}_{\mathrm{e}}^c).$$

As in [9], $\mathrm{P}(\mathcal{E}_{\mathrm{e}}) \to 0$ as $n \to \infty$ if conditions (2), (3), and (4) are fulfilled. We treat $\mathrm{P}(\mathcal{E}_{\mathrm{d}} \,|\, \mathcal{E}_{\mathrm{e}}^c)$ term by term via the union bound. First, note that by the conditional typicality lemma in [6], $\mathrm{P}(\mathcal{E}_0 \,|\, \mathcal{E}_{\mathrm{e}}^c) \to 0$ as $n \to \infty$ (this relies on $\varepsilon' < \varepsilon$). Next, we bound each term $\mathrm{P}(\mathcal{E}_{ijk} \,|\, \mathcal{E}_{\mathrm{e}}^c)$. For each $i \in [5]$, $j \in [3]$, and $k \in [3]$, we show that $\mathrm{P}(\mathcal{E}_{ijk} \,|\, \mathcal{E}_{\mathrm{e}}^c) \to 0$ as $n \to \infty$ is implied by condition (17) with the same indices i, j, and k.

As an example, consider the case of $i = 3$, $j = 3$, $k = 2$.

i	Message subset	m_{10}	m_{12}	m_{13}	m_{11}
1	$\mathcal{M}_{11}$	1	1	1	$\neq 1$
2	$\mathcal{M}_{12}$	1	$\neq 1$	1	any
3	$\mathcal{M}_{13}$	1	1	$\neq 1$	any
4	$\mathcal{M}_{14}$	1	$\neq 1$	$\neq 1$	any
5	$\mathcal{M}_{15}$	$\neq 1$	any	any	any

Table 4. Message subsets $\mathcal{M}_{1i}$.

j	Message subset	m_{20}	l_{21}
1	$\mathcal{M}_{21}$	1	$L_{21}^{(1,1,1)}$
2	$\mathcal{M}_{22}$	1	$\neq L_{21}^{(1,1,1)}$
3	$\mathcal{M}_{23}$	$\neq 1$	any

Table 5. Message subsets $\mathcal{M}_{2j}$.

k	Message subset	m_{30}	l_{31}
1	$\mathcal{M}_{31}$	1	$L_{31}^{(1,1,1)}$
2	$\mathcal{M}_{32}$	1	$\neq L_{31}^{(1,1,1)}$
3	$\mathcal{M}_{33}$	$\neq 1$	any

Table 6. Message subsets $\mathcal{M}_{3j}$.

The probability of the event

$$\mathcal{E}_{332} = \{T(1,1,m_{13},m_{11},m_{20},l_{21},1,l_{31}) \in \mathcal{T}_\varepsilon^{(n)} \text{ for some } m_{13} \neq 1,\, m_{11},\, m_{20} \neq 1,\, l_{21},\, l_{31} \neq L_{31}^{(1,1,1)}\},$$

conditioned on $\mathcal{E}_{\mathrm{e}}^c$, tends to zero as $n \to \infty$ if (18) is true. Recalling the expansion (19), the claim is that $\mathrm{P}(\mathcal{E}_{332} \,|\, \mathcal{E}_{\mathrm{e}}^c) \to 0$ follows from *any* of the following sufficient conditions.

$$\tilde{R}_{13} + R_{11} + R_{20} + \tilde{R}_{21} + \tilde{R}_{31} \le \Diamond, \quad (22a)$$
$$\tilde{R}_{13} + R_{11} + R_{20} + \tilde{R}_{21} + H(X_{31} \,|\, U_3) \le \Diamond, \quad (22b)$$
$$\tilde{R}_{13} + R_{11} + R_{20} + H(X_{21} \,|\, U_2) + \tilde{R}_{31} \le \Diamond, \quad (22c)$$
$$\tilde{R}_{13} + R_{11} + R_{20} + H(X_{21} \,|\, U_2) + H(X_{31} \,|\, U_3) \le \Diamond, \quad (22d)$$
$$\tilde{R}_{13} + R_{11} + H(X_{21}) + \tilde{R}_{31} \le \Diamond, \quad (22e)$$
$$\tilde{R}_{13} + R_{11} + H(S_1 \,|\, U_3) \le \Diamond, \quad (22f)$$

where $\Diamond$ stands for the right-hand side of conditions (18) and (19), namely $H(Y_1 \,|\, U_1, X_{12}, U_3) + I(X_{12}; X_{13} \,|\, U_1)$.

To show that conditions (22a) to (22e) are sufficient, we first bound the probability of the event $\mathcal{E}_{332}$ by omitting S_1^n from the typicality requirement. The subsequent bounding process logically consists of two stages. The first stage treats the signals from transmitters 2 and 3 using methods from interference decoding with point-to-point codes [5]. We use Corollaries 2 and 3, with the indices m_{20}, l_{21}, and l_{31} taking the role of i in the corollaries. Each index can be treated by either one of the corollaries, and varying the division of labor between them leads to the conditions (22a) to (22e). Specifically, if Corollary 2 is used for an index, the corresponding rate term appears in the resulting condition, since the rate term captures the number of possible sequences as counted by their index (in logarithmic scale). On the other hand, using Corollary 3 for an index means that the corresponding entropy term will appear in the condition, as it represents (in logarithmic scale) the number of qualifying sequence realizations in the appropriate typical set. In all cases, the sequences from the first transmitter, X_{13}^n and X_1^n, play the role of dithering random variables. The second stage treats the signals from the desired transmitter. It uses the union bound on the indices l_{13} and m_{11} and borrows from the analysis in disturbance-constrained communication [9]. Finally, to show that condition (22f) is sufficient, we bound the probability of $\mathcal{E}_{332}$ by omitting U_2^n, X_{21}^n, and X_{31}^n from the typicality requirement. Using Corollary 3 for the terms related to transmitters 2 and 3 yields the entropy term $H(S_1|U_3)$.

Similar steps can be followed to show that $\mathrm{P}(\mathcal{E}_{ijk} \,|\, \mathcal{E}_{\mathrm{e}}^c) \to 0$ as $n \to \infty$ for the remaining combinations of (i,j,k). This concludes the proof of Corollary 1.

D. Equivalence of Theorem 1 and Corollary 1

Finally, we show that the regions $\mathscr{R}_{\text{IB}}$ and $\mathscr{R}'_{\text{IB}}$ in Theorem 1 and Corollary 1 are equal. It is clear that $\mathscr{R}_{\text{IB}} \subseteq \mathscr{R}'_{\text{IB}}$, since the conditions of the theorem are more stringent than those of the corollary (compare Remark 7). To show the converse inclusion, we establish that every rate point in $\mathscr{R}'_{\text{IB}}$ is contained in $\mathscr{R}_{\text{IB}}$. The key idea is to vary the auxiliary rates that define the rate split while keeping the overall rate unchanged.

Consider a fixed distribution p. We are given a rate split

$$(R'_{10}, R'_{12}, R'_{13}, \tilde{R}'_{12}, \tilde{R}'_{13}, R'_{11}),$$

which satisfies the conditions of Corollary 1. Define

$$\Delta_{12} = \tilde{R}'_{12} - \min\{\tilde{R}'_{12}, H(X_{12} \,|\, U_1)\},$$
$$\Delta_{13} = \tilde{R}'_{13} - \min\{\tilde{R}'_{13}, H(X_{13} \,|\, U_1)\},$$

and let the modified rate split $(R_{10}, R_{12}, R_{13}, \tilde{R}_{12}, \tilde{R}_{13}, R_{11})$ be given as

$$\begin{aligned} R_{10} &= R'_{10}, \\ R_{12} &= R'_{12} - \Delta_{12}, \\ R_{13} &= R'_{13} - \Delta_{13}, \\ \tilde{R}_{12} &= \tilde{R}'_{12} - \Delta_{12}, \\ \tilde{R}_{13} &= \tilde{R}'_{13} - \Delta_{13}, \\ R_{11} &= R'_{11} + \Delta_{12} + \Delta_{13}. \end{aligned}$$

First note that this rate split maintains the same overall rate $R_{10} + R_{12} + R_{13} + R_{11}$ as the original rate split. To verify non-negativity of each component rate, first note that $R_{10}, \tilde{R}_{12}, \tilde{R}_{13}, R_{11} \geq 0$ by definition. Furthermore,

$$R_{12} = R'_{12} - \tilde{R}'_{12} + \min\{\tilde{R}'_{12}, H(X_{12} \,|\, U_1)\} \geq 0,$$

where the right hand side follows from $R'_{12} \geq 0$ and condition (3). Likewise, it follows that $R_{13} \geq 0$. The modified rate split is thus valid. It remains to be shown that $(R_{10}, R_{12}, R_{13}, \tilde{R}_{12}, \tilde{R}_{13}, R_{11})$ satisfies conditions (2) to (15), using the fact that $(R'_{10}, R'_{12}, R'_{13}, \tilde{R}'_{12}, \tilde{R}'_{13}, R'_{11})$ satisfies (2) to (6) and (17). This is a tedious but straightforward exercise, which we omit here. This concludes the proof that the statements of Theorem 1 and Corollary 1 are equivalent, and thereby, the proof of Theorem 1.

V. Conclusion

The coding scheme that underlies Theorem 1 provides a natural extension of the Han–Kobayashi scheme to interference channels with more than two user pairs. The resulting inner bound strictly includes previously known bounds. This also indicates that the chosen disturbance measure in communication with disturbance constraints is indeed relevant when this setting is embedded in an interference channel.

Finally, although we described the coding scheme and the resulting achievable rate region using the example of deterministic channels with three user pairs, the key ideas generally apply to discrete memoryless interference channels with noise and a larger number of user pairs.

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