

# A Universal Coding Scheme for Remote Generation of Continuous Random Variables

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**Abstract**—We consider a setup in which Alice selects a pdf  $f$  from a set of prescribed pdfs  $\mathcal{P}$  and sends a prefix-free codeword  $W$  to Bob in order to allow him to generate a single instance of the random variable  $X \sim f$ . We describe a universal coding scheme for this setup and establish an upper bound on the expected codeword length when the pdf  $f$  is bounded, orthogonally concave (which includes quasiconcave pdf), and has a finite first absolute moment. A dyadic decomposition scheme is used to express the pdf as a mixture of uniform pdfs over hypercubes. Alice randomly selects a hypercube according to its weight, encodes its position and size into  $W$ , and sends it to Bob who generates  $X$  uniformly over the hypercube. Compared to previous results on channel simulation, our coding scheme applies to any continuous distribution and does not require two-way communication or shared randomness. We apply our coding scheme to classical simulation of quantum entanglement and obtain a better bound on the average codeword length than previously known.

**Index Terms**—Universal code, channel simulation, communication complexity, simulation of quantum entanglement.

## I. INTRODUCTION

Consider the one-shot remote random variable generation setting depicted in Figure 1. Alice and Bob both agree on a set of distributions  $\mathcal{P}$  (over a discrete or continuous set). Alice selects a distribution  $p \in \mathcal{P}$  and wishes to have Bob generate a random variable  $X$  according to this distribution. To accomplish this goal, Alice and Bob use an agreed upon *universal* coding scheme in which Alice uses a stochastic encoder to assign to each  $p \in \mathcal{P}$  a codeword  $W \in \{0, 1\}^*$  from an agreed upon prefix-free code and Bob uses a stochastic decoder to generate a single instance of  $X \sim p$  from the received codeword  $W$ . Let  $L(W)$  be the length of  $W$  in bits. Is there a coding scheme such that for every distribution  $p \in \mathcal{P}$ , Bob can generate  $X \sim p$  with finite expected codeword length  $E_p(L(W))$ ?

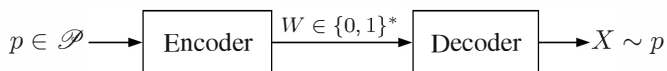


Figure 1. Universal remote generation of random variables.

The answer to this question clearly depends on the set of distributions  $\mathcal{P}$ . Consider the following two simple special cases:

1. Let  $\mathcal{P}$  be the set of probability mass functions over the integers, then we can use the following “generate–compress” strategy. Alice generates  $X \sim p$  and then uses

a universal code over the integers, e.g., [1], [2], to encode  $X$  into  $W$ . Upon receiving  $W$ , Bob recovers  $X$ . Using these codes, the expected codeword length  $E_p(L(W))$  is finite as long as  $E_p(\log X)$  is finite. Note that this scheme uses a stochastic encoder but a deterministic decoder.

2. Let  $X$  be continuous and the class of pdfs  $\mathcal{P}$  has a finite (or countable) cardinality, then we can use the following “compress–generate” strategy. Alice encodes the index of  $p$  into  $W$ . Upon receiving  $W$ , Bob first recovers  $p$  then use it to generate  $X$ . Note that in this scheme, the encoder is deterministic but the decoder is stochastic.

If we index the set  $\mathcal{P}$  by  $\theta \in \Theta$ , then our setting can be viewed as a one-shot synthesis (or simulation) of a channel from  $\theta$  to  $X$  with only one-way communication and without common randomness. Several channel simulation scenarios have been previously studied in classical and quantum information theory. In [3], Bennett et al. considered the asymptotic setting and established the reverse Shannon theorem, which states that  $k$  uses of a channel with capacity  $C$  can be simulated using  $kC + o(k)$  bits of communication with unlimited amount of common randomness. In [4], Winter studied the asymptotic case with limited common randomness and  $\theta_i$  distributed according to a given distribution. He showed that  $kI(\theta; X) + o(k)$  bits of communication and  $kH(X|\theta) + o(k)$  bits of common randomness suffice. Subsequently, Cuff [5] characterized the entire tradeoff region between communication and common randomness for the same setting.

For the one-shot channel simulation setting, schemes based on rejection sampling were developed by Steiner [6], who assumed that Alice and Bob share unlimited common randomness, and by Massar et al. [7], who assumed two-way communication between Alice and Bob. Harsha et al. [8] established a one-shot version of the reverse Shannon theorem using rejection sampling. These rejection sampling schemes, however, are sensitive to the size of  $\mathcal{P}$ —a large size  $\mathcal{P}$  leads to a high rejection rate, which in turn leads to a high computation time.

Note that the aforementioned asymptotic and one-shot channel simulation schemes are not universal since a scheme designed for a channel from  $\theta$  to  $X$  is guaranteed to work only when the simulated distribution lies in the convex hull of the set of output distributions  $\{p(x|\theta)\}$ .

In this paper, we present a universal coding scheme for remote generation of continuous random variables (over scalars or vectors), which we will refer to as *universal dyadic coding*

scheme. When  $\mathcal{P}$  is restricted to the set of orthogonally concave pdfs  $\{f_X(x)\}$ , (which includes quasiconcave), we are able to establish an upper bound on the expected codeword length of  $W$  in terms of  $\sup_x f_X(x)$  and  $\mathbb{E}(\|X\|_\infty)$ . Our scheme uses a dyadic decomposition to express the selected pdf as a mixture of uniform distributions over hypercubes. Alice first selects a hypercube from this mixture at random according to its weight, then encodes its position and size into a codeword  $W$  using an agreed upon universal code over the integers. Upon receiving  $W$ , Bob finds the hypercube and generates  $X$  uniformly over it.

In [9], a similar dyadic decomposition scheme was introduced for distributed simulation of continuous random variables according to an agreed upon pdf in a non-universal manner. In Section II we provide a more detailed comparison between these two dyadic coding schemes.

To further motivate our setup and universal dyadic coding scheme, consider the following application.

**Classical simulation of quantum entanglement.** The simulation of correlations induced by quantum entanglement using classical communication has been widely studied, e.g., see [10], [11], [6]. Consider the Bell state  $|\Phi^+\rangle = (|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)/\sqrt{2}$  of a pair of qubits [12], one held by Alice and the other held by Bob. If Alice measures her qubit in the direction  $\theta_A$  (unknown to Bob) to obtain  $Y_A \in \{\pm 1\}$  and Bob measures his qubit in the direction  $\theta_B$  (unknown to Alice) to obtain  $Y_B \in \{\pm 1\}$ , then  $\mathbb{P}\{Y_A = 1\} = \mathbb{P}\{Y_B = 1\} = 1/2$  and  $\mathbb{E}[Y_A Y_B] = -\cos(\theta_A - \theta_B)$ . By Bell's theorem, it is impossible to simulate the joint distribution of  $(Y_A, Y_B)$  for all  $\theta_A$  and  $\theta_B$  using a classical common randomness source (local hidden variables) between Alice and Bob in place of the qubits. However, such simulation is possible if we instead allow Alice to send a codeword  $W$  to Bob. By a modification of the expression in [13] and letting  $X \in [0, 2\pi]$  be a random variable with conditional pdf

$$f(x|y_A; \theta_A) = \frac{1}{2} \max\{\cos(y_A(x - \theta_A)), 0\}, \quad (1)$$

and  $Y_B = -\text{sgn}(\cos(X - \theta_B))$ , then  $(Y_A, Y_B)$  follows the desired distribution. Hence Alice can generate  $Y_A$  and use our universal remote generation coding scheme to encode  $f(x|Y_A; \theta_A)$  into  $W$  to allow Bob to generate  $X$  and  $Y_B$ . Using Theorem 3 in Section IV, we show that the expected number of bits is bounded as  $\mathbb{E}(L(W)) \leq 12.31$ , and using numerical computation we show that  $\mathbb{E}(L(W)) \leq 8.96$  is achievable. In comparison, the scheme in [7], which also does not require shared randomness, but requires two-way communication (we only allow one-way) provides a looser upper bound of 20 bits on the average number of bits needed.

For clarity of presentation, we first describe our scheme for uniform distributions over subsets of  $\mathbb{R}^n$  and upper bound its expected codeword length. In the Sections III and IV we extend the scheme and bounds to general pdfs and pdfs with uniform bounded support, respectively. The omitted proofs and derivations and a lower bound on the average codeword length can be found in [14].

## A. Notation

Throughout this paper, we assume that log is base 2 and entropy  $H$  is in bits. Log in base  $e$  is written as  $\ln$ . We use the notation:  $[a : b] = [a, b] \cap \mathbb{Z}$ ,

A set  $A \subseteq \mathbb{R}^n$  is orthogonally convex if for any line  $L$  parallel to one of the  $n$  axes,  $L \cap A$  is a connected set (empty, point or interval). A function  $f$  is orthogonally concave if the hypograph  $\{(x, \alpha) : \alpha \leq f(x)\}$  is orthogonally convex.

We denote the volume of a Lebesgue measurable set  $A \subseteq \mathbb{R}^n$  by  $V_n(A) = \int_{\mathbb{R}^n} \mathbf{1}_A(x) dx$ . For  $A, B \subseteq \mathbb{R}^n$ ,  $A+B$  denotes the Minkowski sum  $\{a+b : a \in A, b \in B\}$ , and for  $x \in \mathbb{R}^n$ ,  $A+x = \{a+x : a \in A\}$ . For  $\gamma \in \mathbb{R}$ ,  $\gamma A = \{\gamma a : a \in A\}$ . For  $M \in \mathbb{R}^{n \times n}$ ,  $MA = \{Ma : a \in A\}$ . The erosion  $A \ominus B$  is defined as  $\{x \in \mathbb{R}^n : B+x \subseteq A\}$ .

## II. UNIFORM DISTRIBUTIONS

In this section, we develop our universal dyadic coding scheme for the set of uniform pdfs over finite volume sets  $A \subseteq \mathbb{R}^n$ . We first introduce the dyadic decomposition of a set [9], which is the building block of our coding scheme.

**Definition 1 (Dyadic decomposition).** For  $v \in \mathbb{Z}^n$  and  $k \in \mathbb{Z}$ , define the hypercube  $C_{k,v} = 2^{-k}([0, 1]^n + v) \subset \mathbb{R}^n$ . For a set  $A \subseteq \mathbb{R}^n$  with a boundary of measure zero and  $k \in \mathbb{Z}$ , define the set

$$D_k(A) = \{v \in \mathbb{Z}^n : C_{k,v} \subseteq A \text{ and } C_{k-1, \lfloor v/2 \rfloor} \not\subseteq A\},$$

where  $\lfloor v/2 \rfloor$  is the vector formed by the entries  $\lfloor v_i/2 \rfloor$ . The *dyadic decomposition* of  $A$  is the partitioning of  $A$  into hypercubes  $\{C_{k,v}\}$ ,  $v \in D_k(A)$ ,  $k \in \mathbb{Z}$ . Since every point  $x$  in the interior of  $A$  is contained in some hypercube in  $A$ , the interior is contained in  $\cup_{k \in \mathbb{Z}, v \in D_k(A)} C_{k,v}$ , and the set of points in  $A$  not covered by the hypercubes has measure zero.

Our scheme uses a universal code over the integers to encode the position and size of the hypercubes. In particular, we use the signed Elias delta code defined as follows [2]. Let

$$g_{\gamma+}(k) = 0^N \|1\| a_{N-1} a_{N-2} \dots a_0, \\ g_{\delta+}(k) = g_{\gamma+}(N+1) \|a_{N-1} a_{N-2} \dots a_0\|.$$

Then the signed Elias code is

$$g_\delta(k) = \begin{cases} g_{\delta+}(1-2k) & \text{if } k \leq 0, \\ g_{\delta+}(2k) & \text{if } k > 0, \end{cases}$$

where  $a_N a_{N-1} \dots a_0$  is the binary representation of  $k$ . The length of the codeword  $g_\delta(k)$  is

$$L(g_\delta(k)) = \lfloor \log(2|k|+1) \rfloor + 2 \lfloor \log(\lfloor \log(2|k|+1) \rfloor + 1) \rfloor + 1. \quad (2)$$

We are now ready to define the universal dyadic coding scheme for the set of uniform pdfs.

**Universal dyadic coding scheme for uniform pdfs.** The universal dyadic coding scheme for the set of uniform pdfs over positive, finite volume subsets  $A \subset \mathbb{R}^n$  consists of:

- 1) A stochastic encoder that generates  $\tilde{x}$  according to the observed uniform pdf over  $A$ . It then finds  $w =$

$(k, v)$  such that  $v \in D_k(A)$  and  $\tilde{x} \in C_{k,v}$ . The encoder then maps  $(k, v)$  into a codeword  $w$  which consists of the concatenation of signed Elias delta codewords for  $k, v_1, \dots, v_n$ , i.e.,  $w = g_C(k, v) = g_\delta(k) \| g_\delta(v_1) \| \cdots \| g_\delta(v_n)$ .

- 2) A stochastic decoder that upon receiving  $w$  recovers  $(v, k)$  and generates  $x$  according to a uniform pdf over  $C_{k,v}$ .

The dyadic decomposition for  $\mathbb{R}^2$  and the assignments of codeword to the squares are illustrated in Figure 2. The following illustrates how our scheme is used for a given pdf.

**Example 1.** Consider a uniform pdf over the ellipse  $A = \{x \in \mathbb{R}^2 : x^T K x < 1\}$ ,  $K = \begin{bmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix}$ . Figure 3 depicts the universal dyadic coding scheme for this pdf. The encoder first generates a point in the ellipse uniformly at random, and then sends the codeword representing the square containing the point. The expected codeword length (computed by listing all squares in the dyadic decomposition with side length at least  $2^{-16}$ ) is 15.6. Note that the entropy of  $W$ ,  $H(W) = 6.35$  is significantly smaller since the code is universal.

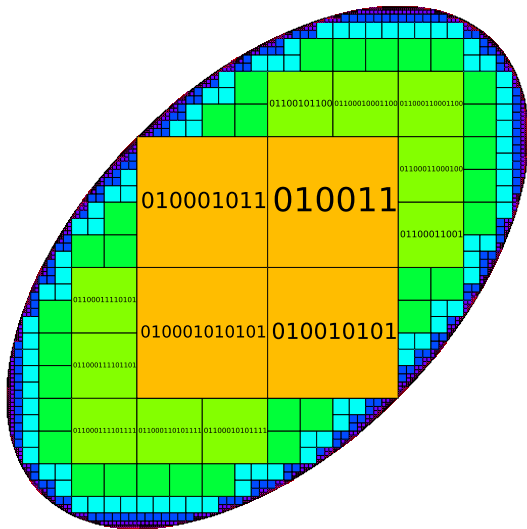


Figure 3. Universal dyadic coding scheme on the uniform distribution over an ellipse.

The length of the codeword of the universal dyadic coding scheme depends on the magnitude of  $k$  and  $v_1, \dots, v_n$ , (which depends on  $k$  and  $\|x\|$ ), hence the length can be bounded using  $k$  and  $\|x\|$ . In [9], it is shown that the expected value of  $k$  can be bounded using the following quantity.

**Definition 2** (Erosion entropy). The *erosion entropy* of the set  $A$  by the set  $B$ , where  $A \subseteq \mathbb{R}^n$  with  $V_n(A) < \infty$ , and  $B \subseteq \mathbb{R}^n$  is a convex set, is defined as

$$h_{\ominus B}(A) = \int_{-\infty}^{\infty} \left( \mathbf{1}\{t \geq 0\} - \frac{V_n(A \ominus 2^{-t}B)}{V_n(A)} \right) dt,$$

where  $A \ominus B = \{x : B + x \subseteq A\}$  is the erosion of  $A$  by  $B$ .

If  $A$  is orthogonally convex, the erosion entropy of  $A$  by the hypercube  $[0, 1]^n$  can be bounded by the expected norm of the uniform distribution on  $A$ , as shown in the following.

**Lemma 1.** Let the set  $A \subseteq \mathbb{R}^n$  be orthogonally convex with  $V_n(A) < \infty$ , and let  $X \sim \text{Unif}(A)$ , then

$$h_{\ominus[0,1]^n}(A) \leq (n-1) \log \mathbf{E}[\|X\|_\infty] - \log V_n(A) + 4n.$$

The proof of this lemma is given in [14]. We now use the erosion entropy to bound the expected codeword length of the universal dyadic coding scheme.

**Theorem 1.** The expected codeword length of the universal dyadic coding scheme for uniform pdfs for  $X \sim \text{Unif}(A)$  is upper bounded as

$$\begin{aligned} \mathbf{E}[L(W)] &\leq n\ell_\delta \left( h + \mathbf{E} \left[ \log \left( \|X\|_\infty + V_n^{1/n}(A) \right) \right] + 4 \right) \\ &\quad + \ell_\delta \left( \log \left( h + 2 \max \left\{ \log V_n^{1/n}(A), 0 \right\} + \frac{5}{2} \right) + 2 \right) \\ &\leq n\ell_\delta (h + \log \mathbf{E}[\|X\|_\infty] + 8) \\ &\quad + \ell_\delta (\log (h + 2 \max \{ \log \mathbf{E}[\|X\|_\infty], 0 \} + 9) + 2), \end{aligned}$$

where  $\ell_\delta(t) = t + 2 \log t$  and  $h = h_{\ominus[0,1]^n}(A)$ .

The proof is given in [14]. Theorem 1 shows that the expected codeword length depends on the erosion entropy, the expected magnitude of  $X$ , and the volume of the set. Intuitively, the erosion entropy measures the complexity of the set (or loosely speaking its surface area to volume ratio). However, the erosion entropy is invariant under shifting. Since our universal scheme is sensitive to the position of  $A$  as well its shape, the bound in Theorem 1 depends also on the expected magnitude of  $X$ . The function  $\ell_\delta(t)$  in Theorem 1 comes from the length of the Elias delta code in (2). Other universal codes for integers may be used in place of Elias delta code, and result in a different bound.

Combining Lemma 1 and Theorem 1, we can bound the expected length of the universal dyadic coding scheme for orthogonally convex sets.

**Corollary 1.** The expected codeword length of the universal dyadic coding scheme for uniform pdfs applied to an orthogonally convex  $A \subseteq \mathbb{R}^n$  is upper bounded as

$$\begin{aligned} \mathbf{E}[L(W)] &\leq n\ell_\delta \left( (n-1) \log r + \log(\|\hat{x}\|_\infty + r) - \log V_n(A) + 4n + 8 \right) \\ &\quad + \ell_\delta \left( \log \left( (n-1) \log r + 2 \max \{r, 0\} \right. \right. \\ &\quad \left. \left. - \log V_n(A) + 4n + 9 \right) + 2 \right). \end{aligned}$$

for any  $\hat{x} \in \mathbb{R}^n$ , where  $\ell_\delta(t) = t + 2 \log t$  and  $r = \mathbf{E}[\|X - \hat{x}\|_\infty]$ .

An added benefit of our universal dyadic coding scheme is that  $X$  can be generated in a distributed manner. Suppose  $X$  is an  $n$ -dimensional vector  $X_1, \dots, X_n$ . Instead of having one decoder wishing to generate  $X$ , we have  $n$  decoders

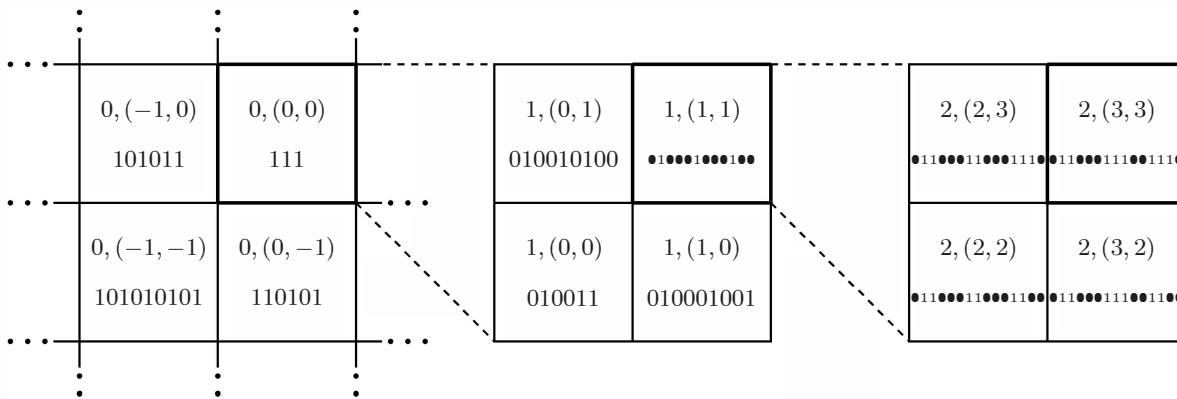


Figure 2. Dyadic squares  $C_{k,v}$  used in the dyadic decomposition for  $n = 2$ , showing their associated  $k, v$  and codeword assignments.

that all receive  $W$  and decoder  $i$  wishes to generate only  $X_i$ ,  $i \in [1 : n]$ . Such distributed generation is possible using our universal dyadic coding scheme since decoder  $i$  can generate  $X_i$  uniformly over the interval  $[2^{-k}v_i, 2^{-k}(v_i + 1)]$  without any need to cooperate with other decoders. In [9], we described a dyadic decomposition coding scheme for distributed generation of a given pdf. The scheme in this paper differs from that in [9] in several aspects.

- The scheme in this paper is universal, while the scheme in [9] is constructed for a given pdf known to both the encoder and the decoders.
- In [9] we used an optimal prefix free code, such as Huffman code, to encode the hypercubes, while in this paper we use a universal code over the integers since the distribution on the hypercubes is not known a priori.
- In [9], we can perform scaling (and bijective transformations) on each variable  $X_i$  before applying the dyadic decomposition scheme. It is not possible to perform such preprocessing here since the decoder would not know the scaling factor or the bijective transformation.
- In the analysis of the expected codeword length in [9], it suffices to consider only the distribution of the sizes of the hypercubes. In our universal scheme, both the size and the position of the hypercube affect the length of the codeword assigned to it.

### III. NON-UNIFORM DISTRIBUTIONS

In this section, we extend the results of the previous section to the case where the pdf of  $X = (X_1, \dots, X_n)$  is selected from a set of arbitrary (not necessarily uniform) pdfs. The key idea in extending our scheme is the following. Note that in general, any pdf can be written as a mixture of uniform pdfs. Let  $Z \sim f_Z$ , where  $f_Z(z) = V_n(L_z^+(f))$  for  $z \geq 0$  and  $L_z^+(f) = \{x \in \mathbb{R}^n : f(x) \geq z\}$  is the superlevel set of  $f$ . Let  $X|Z=z \sim \text{Unif}(L_z^+(f))$ , then we have  $X \sim f(x)$ . Hence  $f(x)$  can be expressed as a mixture of uniform distributions over  $L_z^+(f)$  for different values of  $z$ . Alice can first generate  $Z \sim f_Z$ , then apply the universal dyadic coding scheme for uniform distributions on  $L_z^+(f)$ . The scheme is formally defined as follows.

**Universal dyadic coding scheme for general pdfs.** The universal dyadic coding scheme for the set of almost everywhere continuous pdfs  $\mathcal{P}$  consists of:

- 1) A stochastic encoder that generates  $\tilde{x}$  according to the observed  $f$  and generates  $z$  uniformly in  $[0, f(\tilde{x})]$ , and finds  $(k, v)$  such that  $v \in D_k(L_z^+(f))$  and  $\tilde{x} \in C_{k,v}$ . The encoder maps  $(k, v)$  into a codeword  $w$  that consists of the concatenation of the signed Elias delta codewords for  $k, v_1, \dots, v_n$ , i.e.,  $w = g_C(k, v) = g_\delta(k) \| g_\delta(v_1) \| \dots \| g_\delta(v_n)$ .
- 2) A stochastic decoder that upon receiving  $w$  recovers  $(v, k)$  and generates  $x$  uniformly over  $C_{k,v}$ .

We illustrate this scheme in the following.

**Example 2.** Assume that the selected pdf is Gaussian with zero mean and unit variance. Figure 4 depicts the universal dyadic coding scheme for this pdf. The horizontal and vertical axes represent  $x$  and  $z$ , respectively. The encoder sends the codeword for the rectangle containing  $(x, z)$ . The expected codeword length (computed by listing all intervals in the dyadic decomposition with length at least  $2^{-20}$ ) is 7.06.

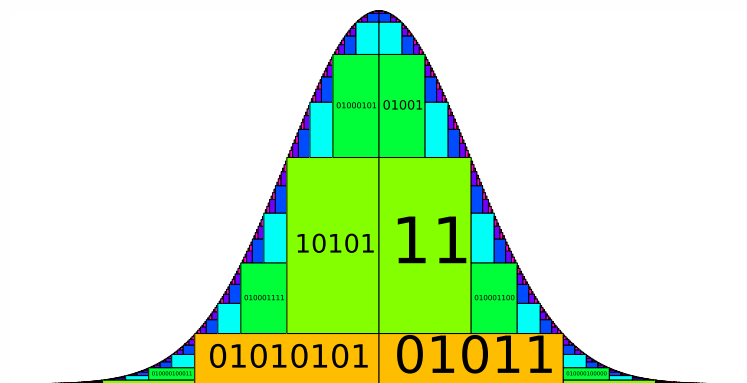


Figure 4. Universal dyadic coding scheme on  $\mathcal{N}(0, 1)$ .

As a consequence of Theorem 1, we have the following bound on the expected codeword length. The proof is given in [14].

**Theorem 2.** *The expected codeword length of the universal dyadic coding scheme for  $X \sim f(x)$  is upper bounded as*

$$\begin{aligned} & \mathbf{E}[L(W)] \\ & \leq n\ell_\delta(\mathbf{E}_Z[h] + \log \mathbf{E}[\|X\|_\infty] + 8) \\ & \quad + \ell_\delta(\log(\mathbf{E}_Z[h] + 2 \max\{\log \mathbf{E}[\|X\|_\infty], 0\} + 10) + 2), \end{aligned}$$

where  $\ell_\delta(t) = t + 2 \log t$  and  $h = h_{\ominus[0,1]^n}(L_z^+(f))$  is a random variable, where  $Z \sim f_Z$ ,  $f_Z(z) = V_n(L_z^+(f))$  for  $z \geq 0$ .

We can also generalize Corollary 1 to orthogonally concave pdfs (which includes quasiconcave pdfs) as follows.

**Corollary 2.** *The expected codeword length of the universal dyadic coding scheme for  $X \sim f(x)$ , where  $f$  is orthogonally concave, is upper bounded as*

$$\begin{aligned} & \mathbf{E}[L(W)] \\ & \leq n\ell_\delta((n-1) \log r + \log(\|\hat{x}\|_\infty + r) + h(Z) + 4n + 8) \\ & \quad + \ell_\delta(\log((n-1) \log r + 2 \max\{\log r, 0\} \\ & \quad + h(Z) + 4n + 10) + 2). \end{aligned}$$

for any  $\hat{x} \in \mathbb{R}^n$ , where  $\ell_\delta(t) = t + 2 \log t$ ,  $r = \mathbf{E}[\|X - \hat{x}\|_\infty]$ ,  $Z \sim f_Z$ ,  $f_Z(z) = V_n(L_z^+(f))$  for  $z \geq 0$ . Note that  $h(Z) \leq \log \sup_x f(x)$ .

#### IV. BOUNDED SUPPORT DISTRIBUTIONS

In this section, we present a variant of the universal dyadic coding scheme for a set of distributions with a uniform bound on their supports. Without loss of generality, assume  $\mathcal{P}$  consists of the set of pdfs over  $[0, 1]^n$ . Since the  $v$  in the definition of our universal dyadic coding scheme (corresponding to the position of the hypercube) is bounded, we can use a fixed length code to encode  $(v_1, \dots, v_n)$ . This allows us to reduce the expected codeword length.

**Universal dyadic coding scheme for pdfs over the unit hypercube.** The universal dyadic coding scheme for pdfs over  $[0, 1]^n$  consists of:

- 1) A stochastic encoder that generates  $\tilde{x}$  according to the observed  $f$  and generates  $z$  uniformly in  $[0, f(\tilde{x})]$ , and finds  $(k, v)$  such that  $v \in D_k(L_z^+(f))$  and  $\tilde{x} \in C_{k,v}$ . The encoder then maps  $(k, v)$  into a codeword which consists of the concatenation of the unsigned Elias gamma codeword for  $k + 1$ , and the  $k$ -bit binary representations of  $v_1, \dots, v_n$ , i.e.,  $w = g_C(k, v) = g_{\gamma+}(k+1) \|g_{b,k}(v_1)\| \cdots \|g_{b,k}(v_n)\|$ , where  $g_{b,k}(i)$  is the binary representation of  $i$  with  $k$  bits, possibly with leading zeros.
- 2) A stochastic decoder that upon observing  $w$  recovers  $(v, k)$  and generates  $x$  uniformly over  $C_{k,v}$ .

Since the length of the unsigned Elias gamma codeword  $g_{\gamma+}(k+1)$  is  $2 \lfloor \log(k+1) \rfloor + 1$ , the length of  $w$  is

$$L(w) = nk + 2 \lfloor \log(k+1) \rfloor + 1.$$

The expected codeword length is upper bounded as follows.

**Theorem 3.** *The expected codeword length of the universal dyadic coding scheme for pdfs over the unit hypercube for  $X \sim f(x)$ , where  $f$  is orthogonally concave, is upper bounded as*

$$\begin{aligned} \mathbf{E}[L(W)] & \leq n(h(Z) + \log n + \log e + 2) \\ & \quad + 2 \log(h(Z) + \log n + \log e + 3) + 1, \end{aligned}$$

where  $Z \sim f_Z$ ,  $f_Z(z) = V_n(L_z^+(f))$  for  $z \geq 0$ . Note that  $h(Z) \leq \log \sup_x f(x)$ .

The proof is given in [14]. As an example, we apply this result to simulating the Bell state (1):

$$f(x | \theta) = \pi \max\{\cos(2\pi(x - \theta)), 0\},$$

fitted to the interval  $[0, 1]$ . Although this pdf is not orthogonally concave, it can be decomposed into at most two orthogonally concave parts with disjoint support, hence the expected codeword length is the weighted average of the expected codeword lengths for those two pieces, which incurs a penalty of at most 1 bit. By Theorem (3),

$$\mathbf{E}[L(W)] \leq \log(\pi e) + 2 \log(\log(\pi e) + 3) + 4 \approx 12.31.$$

We have numerically computed the values of  $\mathbf{E}[L(W)]$  for different  $\theta$ , and verified that  $\mathbf{E}[L(W)] \leq 8.96$  for all  $\theta$ . More details are given in [14].

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