Lecture Notes 9
Spatial Resolution

• Sensor Modulation Transfer Function (MTF)

• MTF Calculation

• Aliasing
Preliminaries

- The image sensor is a spatial (as well as temporal) sampling device (of the incident photon flux image) — the sampling theorem sets the limits for the reproducibility in space (and time) of the input spatial (and temporal) frequencies.

- So spatial (or temporal) frequency components higher than the respective Nyquist rate cannot be reproduced and cause aliasing.

- The image sensor, however, is not a point sampling device in space (or time), and cannot be approximated as such.
  - Photocurrent is integrated over the photodetector area (and in time) before sampling.
  - Photogenerated carriers in quasi-neutral regions of a pixel may diffuse and be collected by its neighboring pixels.

These effects (in addition to the optics) result in *low pass filtering* and *crosstalk* before spatial (and temporal) sampling.
• We focus here on spatial sampling

\[ p \]

\[ \text{Assuming a square pixel with width (pitch) } p, \text{ the spatial Nyquist rate in each dimension is } f_{\text{Nyquist}} = \frac{1}{2p} \text{ and is typically reported in line pairs per millimeter (lp/mm)} \]

• Signals (photon flux images) with spatial frequencies higher than \( f_{\text{Nyquist}} \) cannot be faithfully reproduced, and cause aliasing

• The low pass filtering caused by integration and diffusion degrades the reproduction of frequencies below \( f_{\text{Nyquist}} \) — degradation measured by the Modulation Transfer Function (MTF)
Modulation Transfer Function (MTF)

• The contrast in an image can be characterized by the modulation

\[ M = \frac{S_{\text{max}} - S_{\text{min}}}{S_{\text{max}} + S_{\text{min}}}, \]

where \( S_{\text{max}} \) and \( S_{\text{min}} \) are the maximum and minimum pixel values over the image.

Note that \( 0 \leq M \leq 1 \)

• Let the input to an image sensor be a 1-D sinusoidal monochromatic photon flux

\[ F(x, f) = F_o(1 + \cos(2\pi fx)), \quad \text{for } 0 \leq f \leq f_{\text{Nyquist}} \]

The sensor modulation transfer function is defined as

\[ \text{MTF}(f) = \frac{M_{\text{out}}(f)}{M_{\text{in}}(f)} \]

From the definition of the input signal, \( M_{\text{in}} = 1 \)

• MTF is in general difficult to model and analyze for a real sensor and is determined experimentally.
• By making several simplifying assumptions (as we shall see), the sensor can be modeled as a 1-D linear space-invariant system with impulse response \( h(x) \) that is real, nonnegative, and even.

In this case the transfer function \( H(f) = \mathcal{F}[h(x)] \) is real and even, and the signal at \( x \)

\[
S(x) = F(x, f) * h(x) \\
= F_o(1 + \cos(2\pi fx)) * h(x) \\
= F_o (H(0) + H(f) \cos(2\pi fx))
\]

Therefore

\[
S_{max} = F_o (H(0) + |H(f)|) \\
S_{min} = F_o (H(0) - |H(f)|),
\]

and the sensor MTF is given by

\[
MTF(f) = \frac{|H(f)|}{H(0)}
\]
Simplified MTF Derivation

• We consider a 1-D doubly infinite image sensor

![Diagram of a 1-D doubly infinite image sensor with depletion and quasi-neutral regions.]

• To model the sensor response as a linear space-invariant system
  ○ We assume n+/p-sub photodiode with very shallow junction depth, and therefore we can neglect generation in the isolated n+ regions and only consider generation in the depletion and p-type quasi-neutral regions
  ○ We assume a uniform depletion region (from $-\infty$ to $\infty$)
• The monochromatic input photon flux $F(x)$ to the pixel current $i_{ph}(x)$ can be represented by the linear space invariant system ($i_{ph}(x)$ is sampled at regular intervals $p$ to get the pixel photocurrents)

\[
F(x) \rightarrow d(x) \rightarrow j_{ph}(x) \rightarrow w \cap \left(\frac{x}{w}\right) \rightarrow i_{ph}(x)
\]

Photogeneration Integration

where

\[
\cap \left(\frac{x}{w}\right) = \begin{cases} 
1 & |x| < \frac{w}{2} \\
0 & \text{otherwise,}
\end{cases}
\]

$d(x)$ is the (spatial) impulse response corresponding to the conversion from photon flux to photocurrent density (we will derive it soon), and we assume a square photodetector
• The impulse response of the system is thus given by

\[ h(x) = d(x) \ast w \ast \left(\frac{x}{w}\right) \]

and its Fourier transform (transfer function) is given by

\[ H(f) = D(f)w^2 \text{sinc}(wf), \]

where

\[ \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \]

• Note that \( D(0) = q\text{QE(}\lambda) \), so \( D(f)/q \) can be viewed as a generalized quantum efficiency (function of spatial frequency as well as wavelength)
Finding $D(f)$

- To find $D(f)$, note that the photocurrent density

$$j_{ph}(x) = j_{ph}^{sc}(x) + j_{ph}^{n}(x),$$

where $j_{ph}^{sc}(x)$ is the photocurrent density due to generation in the depletion region and $j_{ph}^{n}(x)$ is the photocurrent density due to generation in the quasi-neutral p-region.

- Assuming an incident 1-D photon flux $F(x)$ (at $z = 0$) with Fourier Transform $F(f)$, then

$$D(f) = \frac{\mathcal{F}[j_{ph}]}{F(f)}$$

so we need to find $\mathcal{F}[j_{ph}]$.

- First note that the generation rate at $(x, z)$ is

$$g(x, z) = \alpha e^{-\alpha z} F(x)$$
• Assuming that all photogenerated charge in the depletion region is collected, we obtain

\[ j_{ph}^{sc}(x) = q \int_0^{L_d} g(x, z) \, dz = qF(x)(1 - e^{-\alpha L_d}) \]

and \( \mathcal{F}[j_{ph}^{sc}] = q(1 - e^{-\alpha L_d})F(f) \)

• Finding \( \mathcal{F}[j_{ph}^{n}] \) is more involved, first recall that

\[ j_{ph}^{n}(x) = qD_n \frac{\partial n_p(x, z)}{\partial z} \bigg|_{z=L_d}, \]

where \( n_p(x, z) \) is the photogenerated electron concentration at \((x, z)\), so

\[ \mathcal{F}[j_{ph}^{n}] = qD_n \frac{\partial N_p(f, z)}{\partial z} \bigg|_{z=L_d} \]

where \( N_p(f, z) \) is the Fourier Transform w.r.t. \( x \) of \( n_p(x, z) \)
• To find $n_p(x, z)$, we need to solve the 2-D continuity equation (in steady state)

$$0 = D_n \left( \frac{\partial^2 n_p}{\partial x^2} + \frac{\partial^2 n_p}{\partial z^2} \right) - \frac{n_p}{\tau_n} + g(x, z),$$

where the $n_p/\tau_n$ term is the recombination (which cannot be ignored here), with the assumed boundary conditions

$$n_p(x, L_d) = 0 \quad \text{(edge of depletion region)}$$
$$n_p(x, L_d + L) = 0 \quad \text{(ohmic contact)}$$

• Since we want to find $N_p(f, z)$, we take the Fourier Transform of the continuity equation w.r.t. $x$ and we obtain

$$0 = D_n \left( (j2\pi f)^2 \cdot N_p(f, z) + \frac{\partial^2 N_p}{\partial z^2} \right) - \frac{N_p}{\tau_n} + G(f, z),$$

where

$$G(f, z) = \mathcal{F}[g(x, z)] = \alpha e^{-\alpha z} F(f)$$
Now we define

$$L_f^2 = \frac{L_n^2}{1 + (2\pi f L_n)^2},$$

where $L_n = \sqrt{D_n \tau_n}$ is the diffusion length of electrons in p-sub (around 600$\mu$m for our 0.5$\mu$m technology).

Substituting in the previous equation we obtain

$$-\frac{\partial^2 N_p}{\partial z^2} + \frac{N_p}{L_f^2} = \frac{G(f, z)}{D_n}$$

The solution w.r.t. $z$ has the form

$$N_p(f, z) = c_1 e^{-\frac{z}{L_f}} + c_2 e^{\frac{z}{L_f}} + c_3 G(f, z)$$

Substituting in the PDE we obtain

$$c_3 = \frac{L_f^2}{D_n (1 - (\alpha L_f)^2)}$$
Using the boundary conditions, we obtain

\[ c_2 = -c_3 \frac{G(f, L_d + L) - G(f, L_d)e^{-\frac{L}{L_f}}}{e^{\frac{L_d+L}{L_f}} - e^{\frac{L_d-L}{L_f}}}, \text{ and} \]

\[ c_1 = -c_2 e^{\frac{L_d}{L_f}} - c_3 G(f, L_d)e^{\frac{L_d}{L_f}} \]

Thus

\[
\mathcal{F}[j_{ph}^n(x)] = qD_n \left( -\frac{c_1}{L_f} e^{-\frac{L_d}{L_f}} + \frac{c_2}{L_f} e^{\frac{L_d}{L_f}} - \alpha c_3 G(f, L_d) \right) \\
= qD_n c_3 \frac{\alpha e^{-\alpha L_d}}{L_f} \left( 1 - \alpha L_f \right) - \frac{e^{-\alpha L} - e^{-\frac{L}{L_f}}}{\sinh \left( \frac{L}{L_f} \right)} \mathcal{F}(f) \\
= qL_f \alpha e^{-\alpha L_d} \left( \frac{1}{1 + \alpha L_f} - \frac{e^{-\alpha L} - e^{-\frac{L}{L_f}}}{(1 - (\alpha L_f)^2) \sinh \left( \frac{L}{L_f} \right)} \right) \mathcal{F}(f)
\]
Combining the results we obtain

\[ F[j_{ph}] = F[j_{ph}^SC(x) + j_{ph}^N(x)] = q \left( 1 - e^{-\alpha L_d} + L_f \alpha e^{-\alpha L_d} \left( \frac{1}{1 + \alpha L_f} - \frac{e^{-\alpha L} - e^{-\frac{L}{L_f}}}{(1 - (\alpha L_f)^2) \sinh(\frac{L}{L_f})} \right) \right) \]

Thus

\[ D(f) = q \frac{1 + \alpha L_f - e^{-\alpha L_d}}{1 + \alpha L_f} - \frac{q L_f \alpha e^{-\alpha L_d} \left( e^{-\alpha L} - e^{-\frac{L}{L_f}} \right)}{(1 - (\alpha L_f)^2) \sinh(\frac{L}{L_f})}, \]

and the system transfer function is given by

\[ H(f) = q \left( \frac{1 + \alpha L_f - e^{-\alpha L_d}}{1 + \alpha L_f} - \frac{L_f \alpha e^{-\alpha L_d} (e^{-\alpha L} - e^{-\frac{L}{L_f}})}{(1 - (\alpha L_f)^2) \sinh(\frac{L}{L_f})} \right) \cdot w^2 \text{sinc}(wf) \]

Finally, the modulation transfer function for \(|f| \leq \frac{1}{2p}\) is

\[ \text{MTF}(f) = \frac{|H(f)|}{H(0)} = \frac{D(f)}{D(0)} \cdot \text{sinc}(wf) \]

- \(\frac{D(f)}{D(0)}\) is called the diffusion MTF
- \(\text{sinc}(wf)\) is called the geometric MTF
Example — Diffusion MTF

For the following examples: \( p = 10\mu m, \quad L_d = 1.8\mu m, \quad \text{and} \quad L = 10\mu m \)
Example — Geometric MTF

MTF

spatial frequency (lp/mm)

w = 4um
w = 6um
w = 8um
Example — Sensor MTF

Here we take $w = 6 \mu m$
Example — Degradation due to MTF

Here $\lambda = 700\text{nm}$
Aliasing occurs when the bandwidth of the input signal exceeds the Nyquist rate – the high frequency components are “folded” into the band. 

\[ S(f) \] refers to the Fourier Transform of the sampled photocurrent for \( |f| \leq \frac{1}{2p} \).
Aliasing Effect

Here we take $\lambda = 700\text{nm}$