Lecture Notes 6
Temporal Noise

- Sources of Noise in Devices
  - Thermal Noise
  - Shot Noise
  - Flicker (1/f) Noise
- Noise Models for Photodiode and MOS Transistor
- Analysis of Noise in PPS
- Analysis of Noise in Photodiode APS
- Active Reset
- Appendix: Background

Introduction

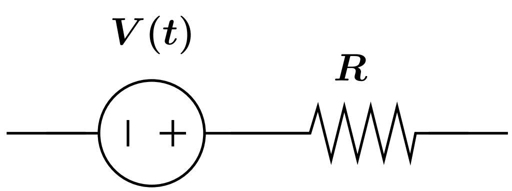
- Temporal noise is the temporal variation in pixel output values under constant illumination due to device noise, supply and substrate noise, and quantization effects
- As we shall see, temporal noise increases with signal (photocurrent), but so does SNR. As a result, its effect is most pronounced at low signal values
- Temporal noise under dark conditions sets a fundamental limit on the sensor *dynamic range*
- Outline:
  - Briefly review sources and models of noise in devices
  - CMOS sensors have more sources of temporal noise than CCDs, and therefore, we focus on analyzing noise in CMOS image sensors
  - Noise in CCDs is covered by a subset of our analysis
- Our analysis approach and results are quite general and apply to many other types of circuits

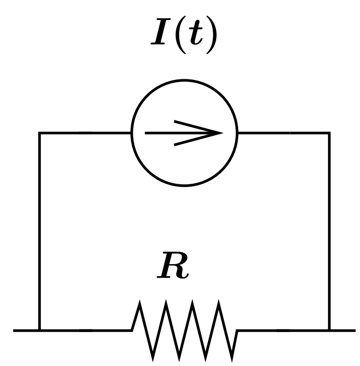
Noise Sources in Devices — Thermal Noise

- *Thermal noise* is generated by thermally induced motion of electrons in conductive regions, e.g., carbon resistors, polysilicon resistors, MOS transistor channel in strong inversion
- It has zero mean, very flat and wide bandwidth (GHzs) psd, and is Gaussian – modeled as WGN voltage/current source with zero mean and power spectral density

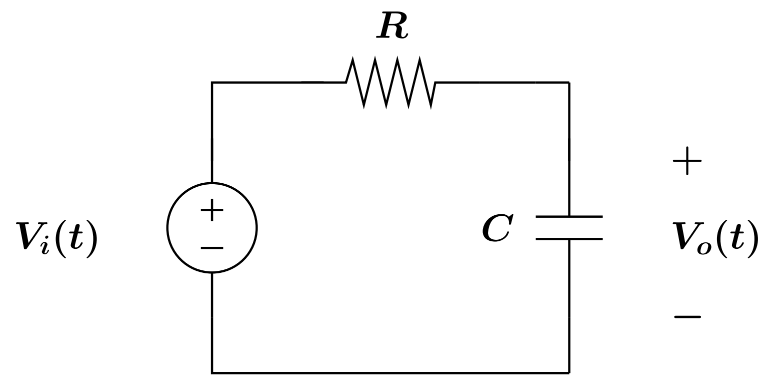
$$S_V(f) = 2kTR \text{ V}^2/\text{Hz} \ / \quad S_I(f) = \frac{2kT}{R} \text{ A}^2/\text{Hz} \quad \text{for all } f,$$

respectively, where k is the Boltzman constant and T is the temperature in Kelvin, in series/parallel with the resistor





- kT/C noise: Consider an RC circuit in *steady state*



$V_i(t)$ is the thermal noise associated with R and is a WGN process with $S_{V_i}(f) = 2kTR$. Find the average output power $\overline{V_o^2(t)}$

Solution: Since the circuit is assumed to be in steady state, we can use frequency domain analysis. The transfer function from V_i to V_o is

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

Thus

$$|H(f)|^2 = \frac{1}{1 + (2\pi fRC)^2},$$

The output voltage psd is

$$S_{V_o}(f) = S_{V_i}(f)|H(f)|^2 = 2kTR\frac{1}{1 + (2\pi fRC)^2}$$

Thus the average output power is

$$\begin{aligned}\overline{V_o^2(t)} &= \int_{-\infty}^{\infty} S_{V_o}(f)\, df \\ &= \int_{-\infty}^{\infty} \frac{2kTR\, df}{1 + (2\pi f RC)^2}\end{aligned}$$

Now define $x = 2\pi f RC$ and use the fact that

$$\int_{-a}^{a} \frac{dx}{1 + x^2} = 2 \arctan a$$

to obtain

$$\begin{aligned}\overline{V_o^2(t)} &= \frac{kT}{\pi C} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} \\ &= \frac{kT}{\pi C} \arctan(x)\Big|_{-\infty}^{\infty} \\ &= \frac{kT}{C}\end{aligned}$$

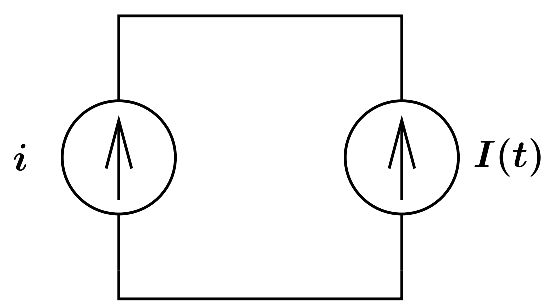
Note that the result is independent of R !!

This is a *very* important result that is commonly used (correctly and incorrectly) in the analysis of noise in circuits

Noise Sources in Devices — Shot Noise

- Generated by fluctuations in static (dc) current flow through depleted regions, e.g., in a pn junction diode, a bipolar transistor, or MOS transistor in subthreshold regime
- Each carrier crossing the edge of the depletion region can be considered as a purely random event and independent of all other carriers. Therefore, the current can be modeled as a Poisson point process with high rate
- We can thus approximate the current as a dc source i (representing the mean of the Poisson process) in parallel with a WGN current source (limit of Poisson with high rate) with zero mean and psd

$$S_I(f) = qi \text{ A}^2/\text{Hz} \quad \text{for all } f, \text{ where } q \text{ is the electron charge}$$

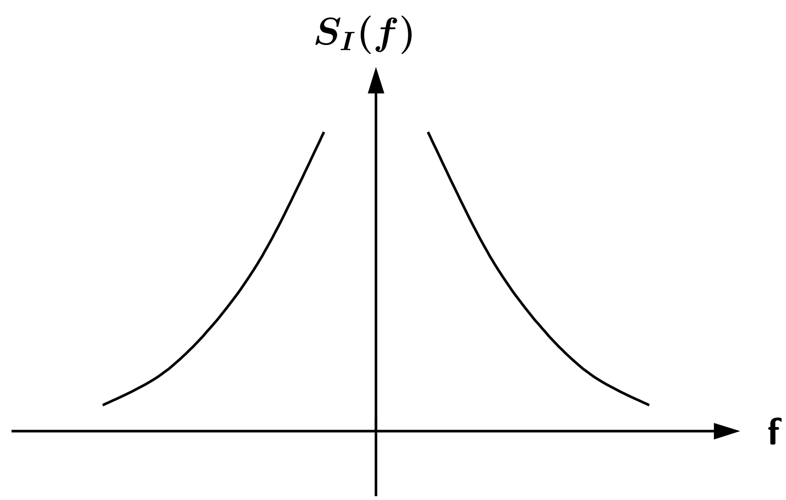


Noise Sources in Devices — Flicker Noise

- Flicker noise is associated with static (dc) current flow in both conductive and depletion regions. It is caused by surface and bulk traps due to defects and contaminants that randomly capture and release carriers
- Flicker noise has zero mean and psd that falls off with f (since some of the time constants associated with the capture and release of carriers are relatively long)

$$S_I(f) \propto i^c \frac{1}{|f|^n}, \text{ for } \ 0.5 \leq c \leq 2$$

- For $n = 1$, it is called 1/f noise



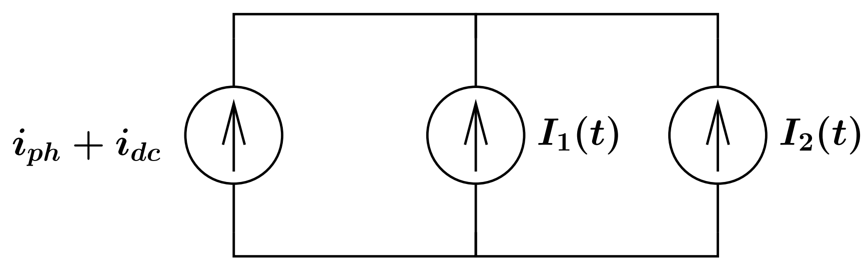
- 1/f noise is most significant at low frequencies below MHz range
- Samples of 1/f noise can be highly correlated
- 1/f noise is not really a stationary process (since the more we wait the more 1/f noise we see) – it is usually approximated by a stationary process with 1/f psd within a frequency range $[f_{\min}, f_{\max}]$
    - $f_{\min}$ is determined by the circuit "observation time"
    - $f_{\max}$ is where 1/f noise psd is sufficiently lower than thermal/shot noise

Photodiode Noise Model

- The dominant source of noise in a photodiode is shot noise due to photo and dark currents. Flicker noise can become somewhat significant when integration time is very long
- Noise represented by two current sources in parallel with zero means and psds

$$S_{I_1}(f) = q(i_{ph} + i_{dc}), \text{ for all } f$$
$$S_{I_2}(f) = a\frac{(i_{dc} + i_{ph})^c}{|f|}, \text{ for } |f| \in [f_{min}, f_{max}],$$

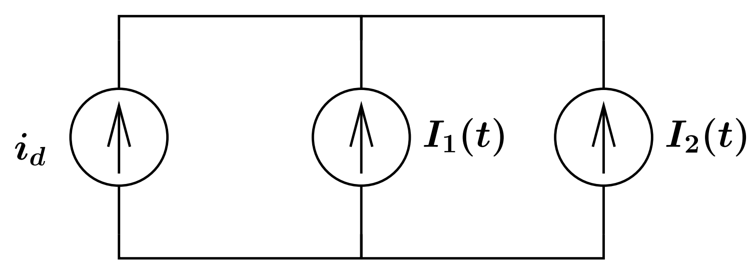
where $0.5 \leq c \leq 2$ and a is a constant that depends on the physical characteristics of the diode



- The two noise sources are statistically independent

MOS Transistor Noise Model

- The dominant source of noise in an MOS transistor is thermal noise, since the MOS transistor channel in strong inversion (i.e., when it is ON) is conductive
- In subthreshold, i.e., for $0 < v_{gs} < v_T$, the dominant source of noise is shot noise (the operation is similar to a bipolar transistor)
- MOS transistors also suffer from flicker noise (due to slow traps in the gate oxide)
- Noise modeled by two statistically independent current sources, $I_1(t)$ for the thermal (or shot) noise and $I_2(t)$ for the flicker noise, in parallel with the drain current i_d



- The thermal (or shot) noise source is modeled as WGN with zero mean and psd

$$S_{I_1}(f) = \begin{cases} \frac{4}{3}kTg_m, & \text{in saturation (thermal)} \\ 2kT/R_{ds} & \text{in ohmic (thermal)} \\ qi_d & \text{in subthreshold (shot),} \end{cases}$$

where

  - g_m is the MOS transistor small signal transconductance

$$g_m = \left.\frac{\partial i_d}{\partial v_{gs}}\right|_{\substack{v_{ds} = v_{ds0} \\ v_{gs} = v_{gs0}}} = k_{n,p}\frac{W}{L}(v_{gs0} - v_T)(1 + \lambda v_{ds0}) \text{ mho},$$

    where v_T is the transistor threshold voltage

  - R_{ds} is the transistor (source to drain) resistance in the linear region

$$R_{ds} = \left.\frac{\partial v_{ds}}{\partial i_d}\right|_{v_{gs}=v_{gs0}} \approx \left(k_{n,p}\frac{W}{L}(v_{gs0} - v_T)\right)^{-1},$$

    provided $v_{ds0} \ll (v_{gs0} - v_T)$

- In the subthreshold region, i.e., for $0 < v_{gs} < v_T$,

$$i_d = \frac{W}{L} i_0 e^{\frac{\kappa v_{gs} - (1-\kappa) v_{sb}}{v_t}} (1 - e^{-\frac{v_{ds}}{v_t}}),$$

  where

  v_{gs} is the gate to source voltage,

  v_{ds} is the drain to source voltage,

  v_{sb} is the source to bulk voltage,

  $\kappa = \frac{C_{ox}}{C_{ox} + C_{depletion}}$ is the *gate efficiency factor*,

  $v_t = \frac{kT}{q}$ is the *thermal voltage*, and

  i_0 is a constant that depends on the transistor threshold voltage

- The flicker noise source $I_2(t)$ has zero mean and psd

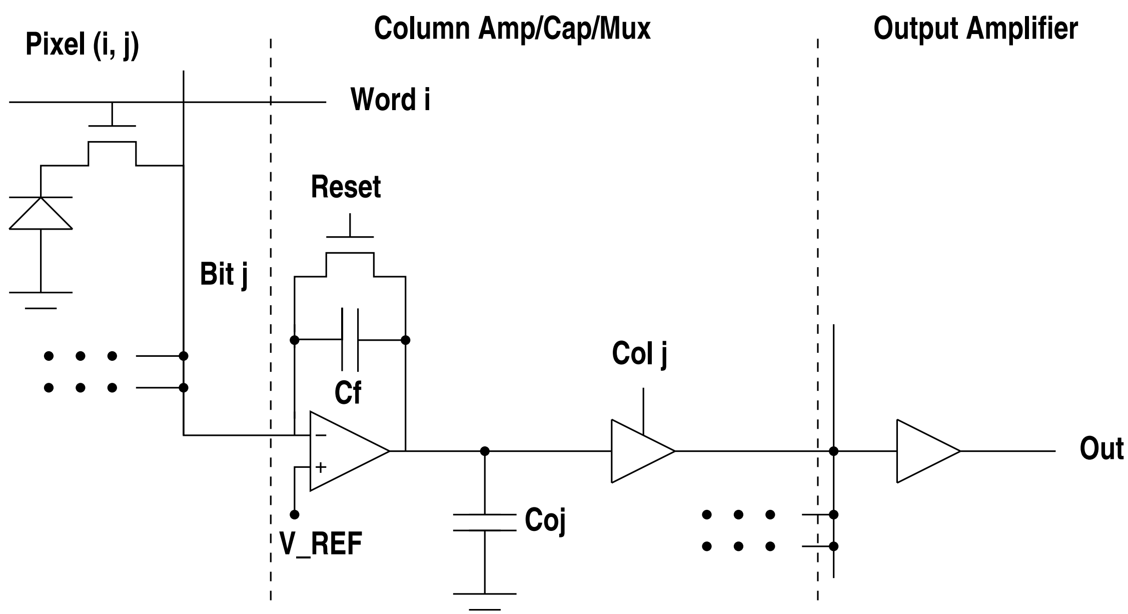
$$S_{I_2}(f) = a \frac{i_d^c}{|f|} \text{ A}^2/\text{Hz},$$

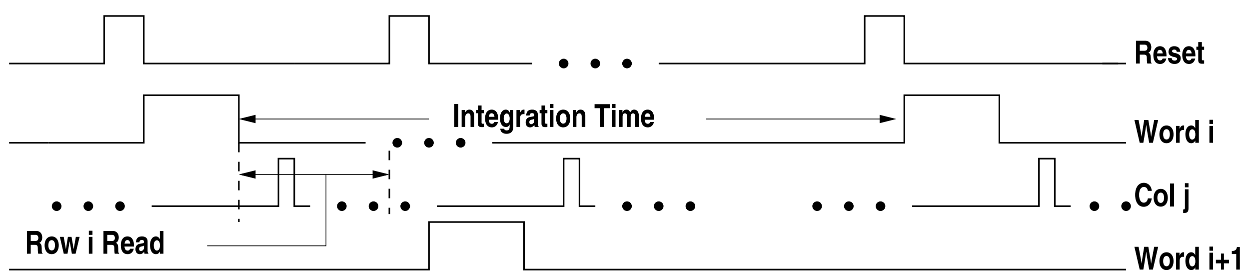
where a depends on the physical characteristics of the transistor

Analysis of Noise in Image Sensors

- Temporal noise in an image sensors is due to:
  - Shot noise of photodetector photo and dark currents
  - Thermal/shot noise of transistors used for readout and amplification
  - 1/f noise of photodetector and transistors
  - Substrate and supply voltage noise
  - Quantization noise
- The analysis of noise due to the phototdetector applies to all image sensors operated in direct integration
- Analysis of noise in CCDs is simpler than in CMOS image sensors – CMOS sensors have more transistors in the signal path (access transistors, pixel, and column amplifiers) versus a single output amplifier for CCDs
- Our analysis will not include 1/f noise (more complex due to high correlation), substrate and supply noise (difficult to quantify) and quantization noise (can be added at the end)

Analysis of Noise in Passive Pixel Sensor (PPS)





- We aim to find the rms value of the input referred noise at the end of readout
- First, we list the sources of noise in PPS that we will consider during each phase of its operation

  During integration:

  - Shot noise due to photo and dark current of photodiode

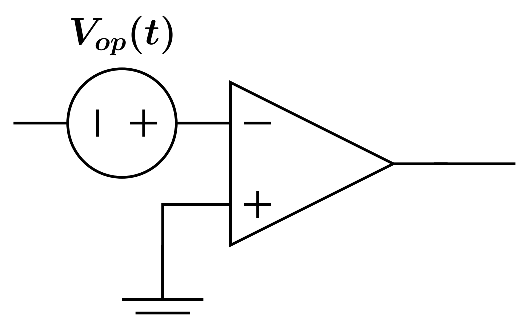
  During reset:

  - Thermal noise due to reset transistor
  - Opamp noise

  During readout:

  - Thermal noise due to access transistor
  - Opamp noise
  - Noise fed back to the photodiode by the above sources
  - Noise due to column and chip amplifiers
  - Noise due to photo and dark current

- To compute the noise we analyze each phase of the operation separately
  - We find the average output referred noise power at the end of each stage
  - Since the noise contributions can be assumed independent, we add them up to find the total average output noise power – here we neglect any decay of noise samples before the end of readout
  - We then find the equivalent input referred noise
- We model the input referred charge amplifier opamp noise as white noise voltage source $V_{op}(t)$ with psd $N/2$, for all f



- We ignore column and chip follower amplifier noise (not significant compared to other source) and photo and dark current during readout (homework problem)

- For numerical examples, we assume

  $C_D = C_f = 20fF$

  $C_b = 0.6656pF$

  For the opamp $A = 6 \times 10^4$, $\omega_o = 100$rad/s, and $N = 10^{-14}$ V^2/Hz

  ON resistance of reset and access transistors $R_s = 10$kohm

  Integration time $t_{int} = 30$ms

  Dark current $i_{dc} = 5$fA

Output Noise Due to Integration

- During integration shot noise due to photo and dark currents is integrated over the photodiode capacitance
- The mean square of the noise charge at the end of integration is given by

$$\begin{aligned}
\overline{Q^2_{\text{shot}}(t_{int})} &= \int_0^{t_{int}}\int_0^{t_{int}} \overline{I(t_1)I(t_2)}\,dt_1\,dt_2 \\
&= \int_0^{t_{int}}\int_0^{t_{int}} q(i_{ph}+i_{dc})\delta(t_1-t_2)dt_1dt_2 \\
&= q(i_{ph}+i_{dc})t_{int}\,\text{Col}^2
\end{aligned}$$

- The output noise charge $Q_{\text{shot}}(t_{int})$ is transferred to the output during readout and the resulting average output noise power

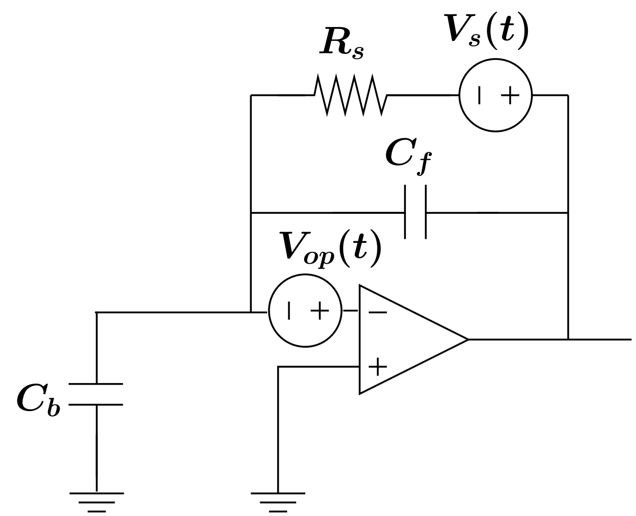
$$\overline{V^2_{o1}} = \frac{q}{C_f^2}(i_{ph}+i_{dc})t_{int}\,\text{V}^2$$

- Using the provided numbers, we obtain

$$\overline{V^2_{o1}} = \begin{cases} 6\times 10^{-8}\text{V}^2 & \text{for } i_{ph}=0 \\ 210\times 10^{-8}\text{V}^2 & \text{for } i_{ph}=170\text{fA} \end{cases}$$

Output Noise Due to Reset

- The sensor circuit including the noise sources during reset:



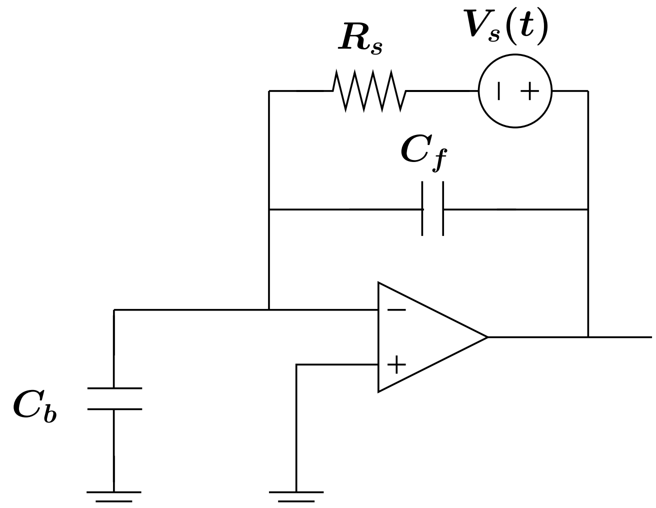
The psds for the noise sources are

$$S_{V_s}(f) = 2kTR_s, \text{ for all } f,$$
$$S_{V_{op}}(f) = \frac{N}{2}, \text{ for all } f$$

- To find the average output noise, we calculate it for each source and use superposition

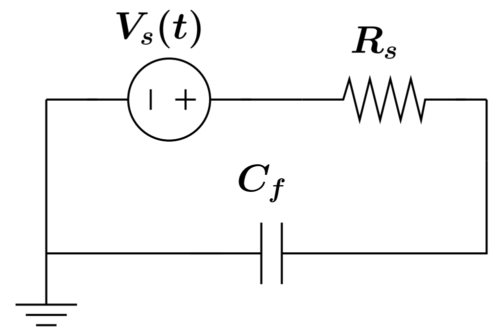
Output Noise Due to Reset Transistor

- Here we only consider the reset transistor noise



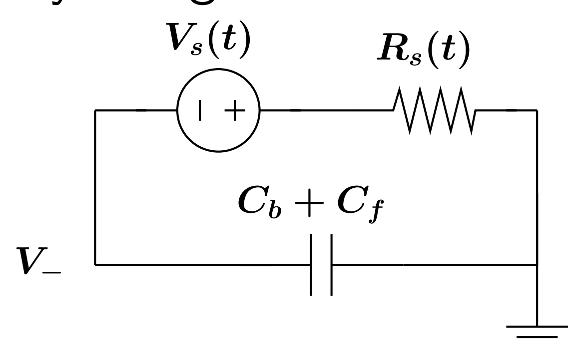
- At the end of reset, noise is stored on C_b and C_f, the sampled noise is then transferred to the output at the end of readout

- First we consider two extreme cases
  - Case 1: opamp has infinite gain-bandwidth



In this case, no noise is stored on C_b and the noise stored on C_f has average power $\frac{kT}{C_f} = 20.7 \times 10^{-8}\text{V}^2$ and is transferred to the output during readout

  - Case 2: opamp has very low gain-bandwidth



In this case noise is stored on both C_b and C_f, and its average power is $\frac{kT}{(C_b+C_f)}$

During readout the noise stored on C_b is amplified, resulting in output noise power $\frac{kT}{(C_b+C_f)}\left(\frac{C_b}{C_f}\right)^2 \approx \frac{kT}{C_f}\left(\frac{C_b}{C_f}\right) = 689 \times 10^{-8}\text{V}^2$, which is much larger than in the previous case

- We are not sure which case holds, so we first find the transfer function

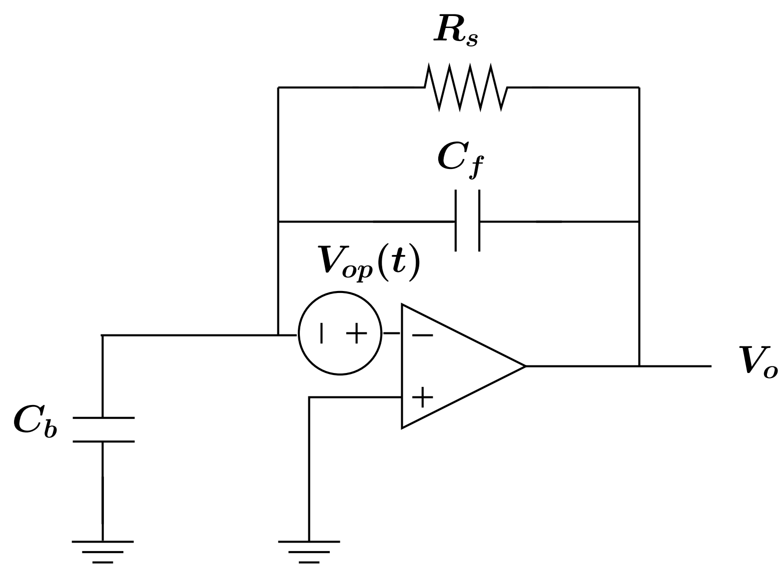
$$\frac{V_-(f)}{V_s(f)} \approx \frac{j\left(\frac{f}{Af_0}\right)}{1 + jf\frac{1}{Af_o} + j2\pi f R_s C_f - 2\pi f^2 \frac{(C_b+C_f)R_s}{Af_o}}$$

We then numerically determine the average output noise

$$\overline{V_{o2}^2} = 2kTR_s \int_{-\infty}^{\infty} \left|\frac{V_-(f)}{V_s(f)}\right|^2 df \times \left(\frac{C_b}{C_f}\right)^2 = 669 \times 10^{-8}\ \text{V}^2,$$

which, unfortunately, is very close to the second (worse) case

Output Reset Noise Due to Opamp



- To find the noise sampled on C_b, we first find the transfer function

$$\frac{V_b(f)}{V_{op}(f)} = -\frac{1 + jf/f_2}{1 - \frac{f^2}{Af_o f_1} + jf\left(\frac{1}{f_2} + \frac{1}{Af_1} + \frac{1}{Af_o}\right)},$$

where $f_1 = \frac{1}{2\pi(C_b+C_f)R_s}$ and $f_2 = \frac{1}{2\pi C_f R_s}$

- The average noise power stored on C_b is thus given by

$$\begin{aligned}\overline{V_b^2} &= \frac{N}{2}\int_{-\infty}^{\infty}\left|\frac{V_b(f)}{V_{op}(f)}\right|^2 df \\ &= 1.5 \times 10^{-8}\ \mathsf{V}^2\end{aligned}$$

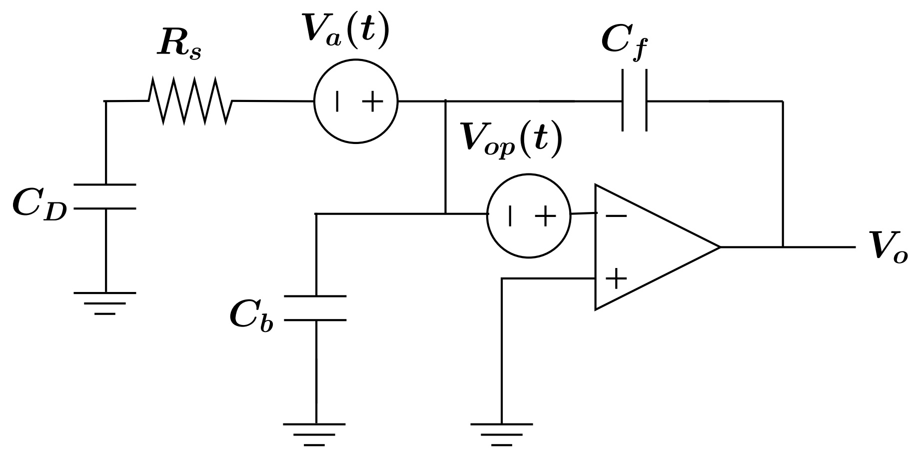
- This noise is amplified during readout and we obtain

$$\overline{V_{o3}^2} = \left(\frac{C_b}{C_f}\right)^2 \times \overline{V_b^2} = 1661 \times 10^{-8}\ \mathsf{V}^2$$

- Note that this noise is even larger than the one due to the reset transistor!

Output Noise Due to Readout

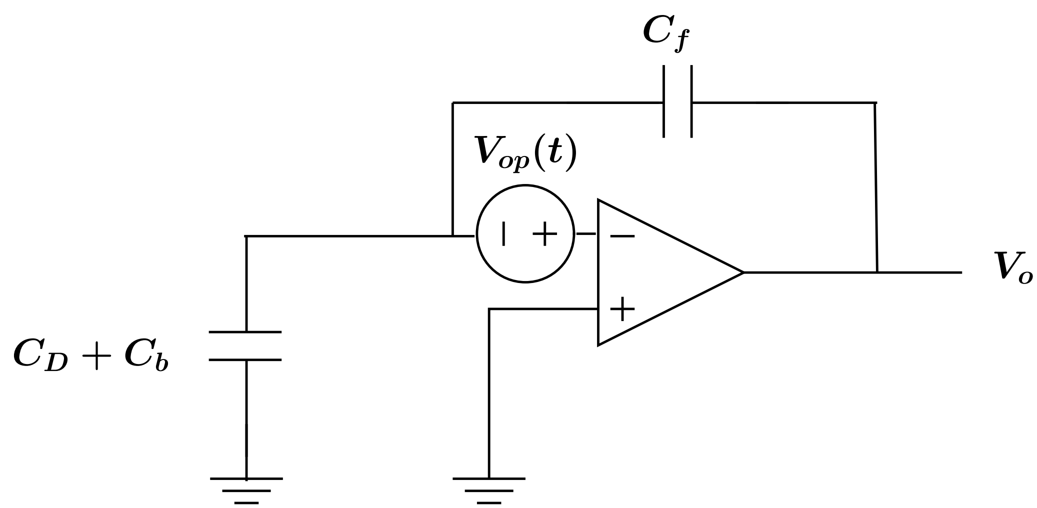
- Consider the PPS circuit during readout including the noise sources



- The two noise sources are zero mean and uncorrelated
- These sources also result in noise stored on C_D at the end of readout, which is read out during the following readout (only relevant in video operation)
- Readout noise due to the access transistor can be shown to be very small and therfore we neglect it

Output Readout Noise Due to Opamp

- We ignore R_s here to simplify the analysis (doesn't affect the result much)



- The transfer function is

$$\frac{V_o(f)}{V_{op}(f)} = -\frac{C_b + C_D + C_f}{C_f} \cdot \frac{1}{1 + j\frac{f}{\beta}},$$

where

$$\beta = \frac{Af_oC_f}{C_f + C_D + C_b}$$

- The average output noise is given by

$$\overline{V_{o4}^2} = \frac{N}{2}\left(\frac{C_b + C_D + C_f}{C_f}\right)^2 \int_{-\infty}^{\infty} \frac{df}{1 + (\frac{f}{\beta})^2}$$

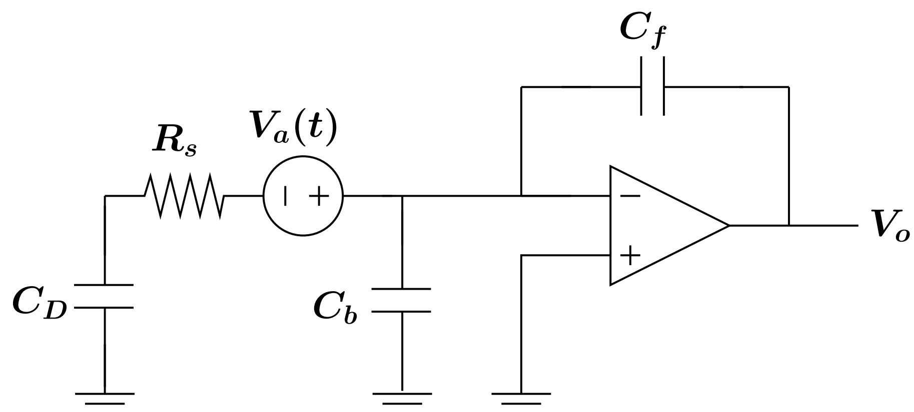
$$= \frac{N}{2}\left(\frac{C_b + C_D + C_f}{C_f}\right)^2 \beta\pi \ \mathsf{V}^2$$

- For the given parameters

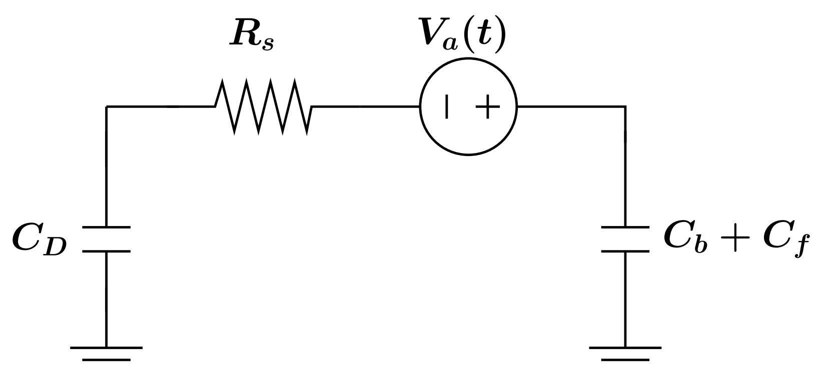
$$\overline{V_{o4}^2} = 52.9 \times 10^{-8} \ \mathsf{V}^2$$

Noise Fed Back to the Photodiode

- At the end of readout, the two noise sources store noise on the photodiode. This noise is read out during the following readout
- Noise due to the access transistor



We approximate the circuit (bad opamp) as



This yields average noise power on C_D of

$$\overline{V_{C_D}^2(t)} \approx \frac{kT}{C_D}$$

This is added to the average output noise power as

$$\overline{V_{o5}^2} = \frac{kT}{C_D}\left(\frac{C_D}{C_f}\right)^2 = 20.7 \times 10^{-8}\ \mathsf{V}^2$$

- Noise due to the opamp is very small and can be neglected

Total Average Noise Power

- First we summarize the results
  - During integration:

$$\overline{V_{o1}^2} = \begin{cases} 6 \times 10^{-8}\text{V}^2 & \text{for } i_{ph} = 0 \\ 210 \times 10^{-8}\text{V}^2 & \text{for } i_{ph} = 170\text{fA} \end{cases}$$

    $\overline{V_{o1}^2}$ depends on photocurrent, dark current, and integration time t_{int}, i.e., is signal dependent

  - During reset due to reset transistor:

$$\overline{V_{o2}^2} = 669 \times 10^{-8}\ \text{V}^2$$

  - During reset due to opamp:

$$\overline{V_{o3}^2} = 1661 \times 10^{-8}\ \text{V}^2$$

    Note these noise components do not depend on photo and dark current or on integration time

- ◦ During readout due to opamp:

$$\overline{V_{o4}^2} = 52.9 \times 10^{-8}\text{V}^2$$

- ◦ Noise fed back to photodiode during readout:

$$\overline{V_{o5}^2} = 20.7 \times 10^{-8}\ \text{V}^2$$

  Note that these noise compoenents also do not depend on photocurrent, dark current, or integration time (noise due to photo and dark current during readout is small)

- The total output referred average noise power is given by

$$\overline{V_o^2} = \sum_{i=1}^{5} \overline{V_{oi}^2}$$

  Adding up the terms, we obtain

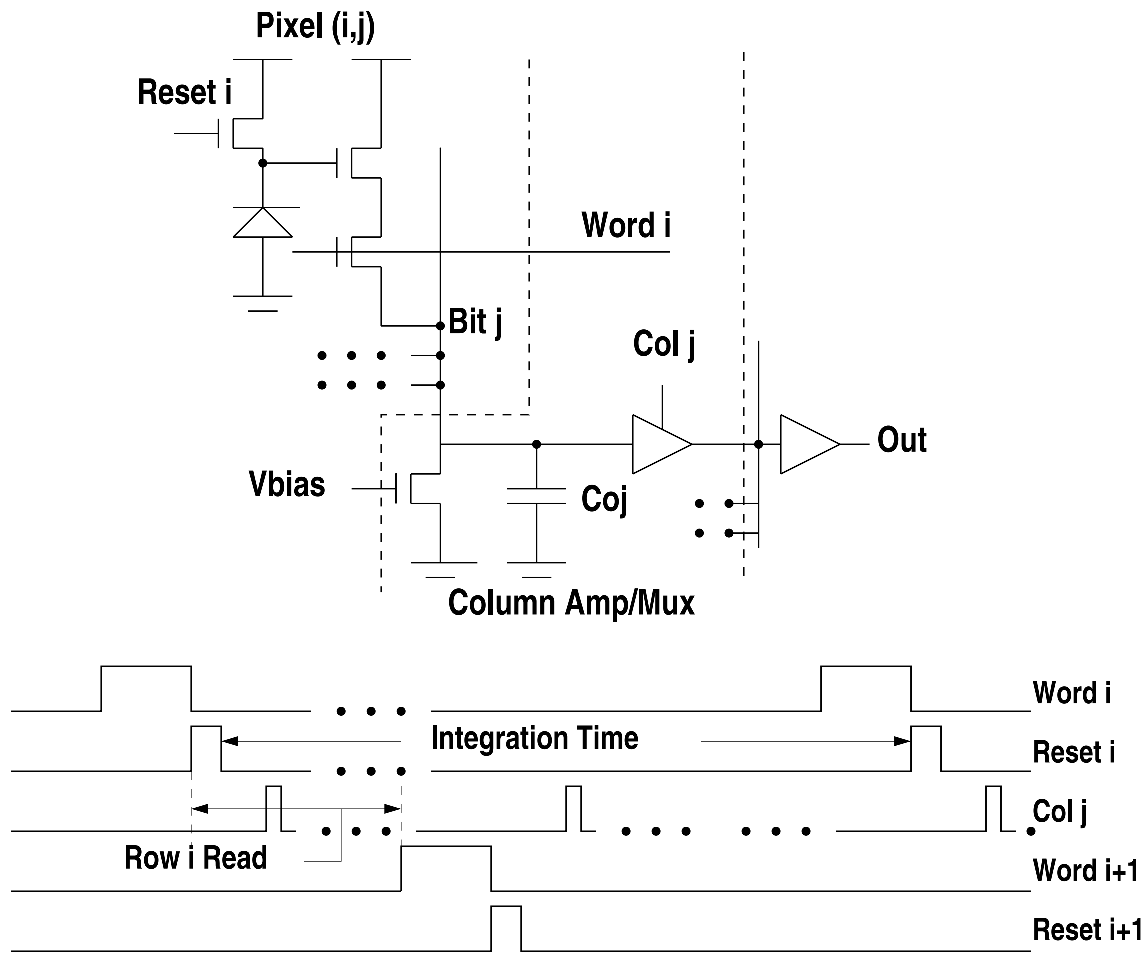
$$\overline{V_o^2} = \begin{cases} 2409.6 \times 10^{-8}\text{V}^2 & \text{for } i_{ph} = 0 \\ 2613.6 \times 10^{-8}\text{V}^2 & \text{for } i_{ph} = 170\text{fA} \end{cases}$$

- The output rms value of the noise

$$\text{rms}(V_o) = \sqrt{\overline{V_o^2}} = \begin{cases} 4.91\text{mV} & \text{for } i_{ph} = 0 \\ 5.11\text{mV} & \text{for } i_{ph} = 170\text{fA} \end{cases}$$

- The total average noise power is typically reported as input refrerred charge in electrons
  To find it we simply multiply the rms output voltage by the reciprocal of the conversion gain (C_f/q), which gives 613 electrons for $i_{ph} = 0$ and 638 electrons for $i_{ph} = 170$fA (this is huge!!)

Analysis of Noise in Active Pixel Sensor (APS)



- We consider the following sources of noise

  During integration:

  - Shot noise due to photodiode photo and dark currents

  During reset:

  - Thermal(?) noise due to reset transistor, and shot noise due to photodiode photo and dark current

  During readout:

  - Thermal noise due to the follower transistor
  - Thermal noise due to the access transistor
  - Thermal noise due to the bias transistor

- To find the average output noise power at the end of readout, we find the average output noise power due to each stage of the operation and sum them up
- The output noise due to integration can be found in the same way as for the PPS, and we obtain

$$\overline{V_{o1}^2} = \frac{q}{C_D^2}(i_{ph} + i_{dc})t_{int}\ \mathsf{V}^2,$$

assuming that the follower amplifier has unity gain

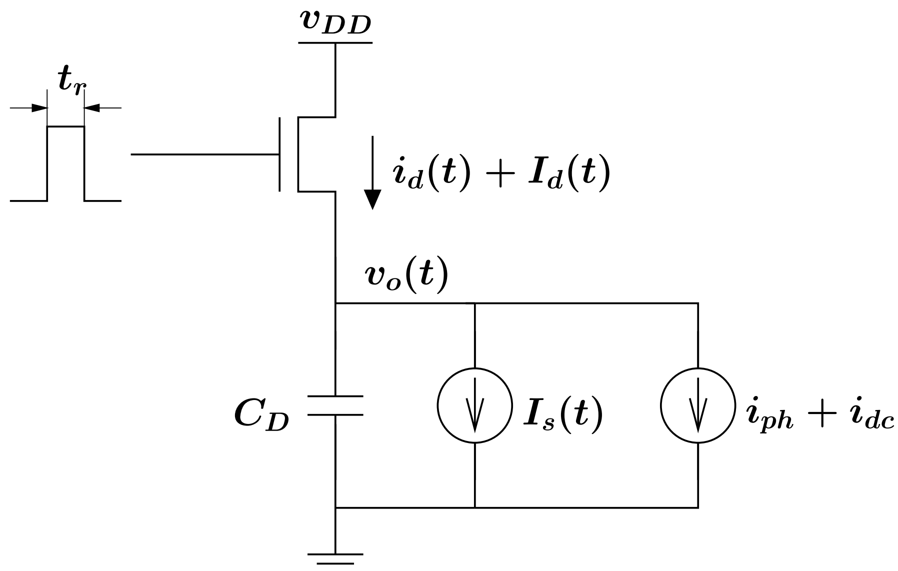
Reset Noise

- Noise depends on the type of reset used
- Hard reset:
  - Here the gate of the reset transistor is biased high enough to reset the diode to v_{DD} (or whatever voltage the drain is set to)
  - In this case, the reset transistor operates in the ohmic region and we have a simple RC circuit with a thermal noise voltage source with psd of $2kTR_{ds}$
  - The average noise power at the end of integration (ignoring shot noise due to photo and dark current) is given by

$$\overline{V_o^2} = \frac{kT}{C_D}$$

Analysis of Soft Reset Noise [Tian et al.'01]

- During soft reset the gate of the reset transisor is biased to v_{DD}
- The noise is due to the drain current $i_d(t)$ and the photodiode photo and dark currents



- The noise current $I_d(t)$ is associated with the drain current $i_d(t)$ and the noise source $I_s(t)$ is the shot noise due to the photodiode photo and dark currents
- The reset transistor reaches subthreshold very quickly (within 1ns)

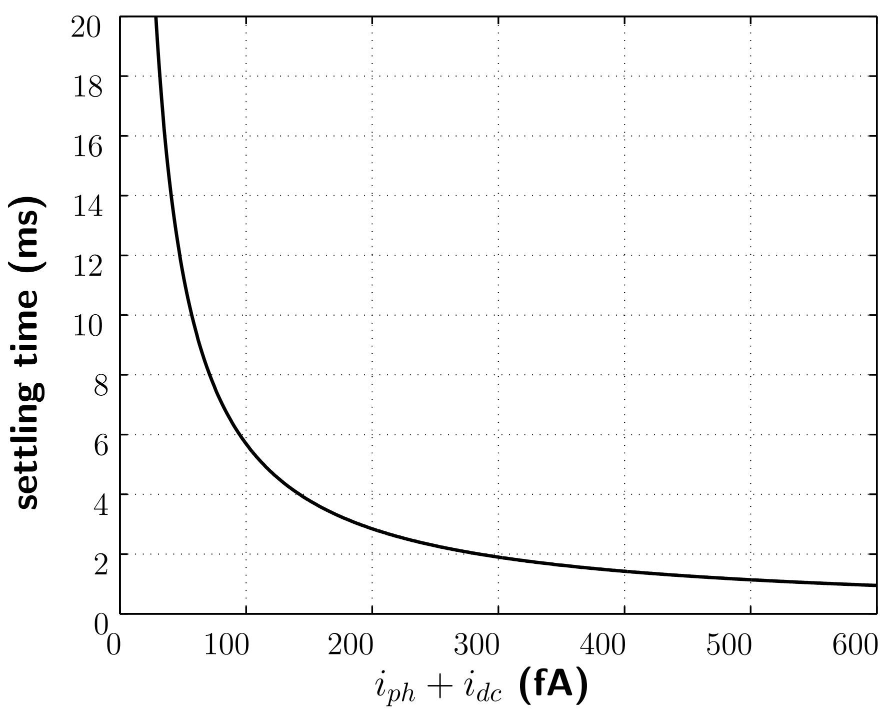
- In subthreshold $I_d(t)$ is due to shot noise and has psd

$$S_{I_d}(f) = q i_d(t) \text{ A}^2/\text{Hz}$$

- If the circuit is in reset long enough, it reaches steady state (where $i_d = i_{ph} + i_{dc}$) and the average noise power is simply

$$\overline{V_o^2} = \frac{kT}{C_D}$$

- Unfortunately (or fortunately) the circuit does not have enough time to reach steady state



- From the above graph, we see that the circuit needs > 1ms to achieve steady state, which is much longer than typical reset time

- Analyzing the circuit before it reaches steady state is beyond the scope of the course, and we only give the answer

$$\overline{V_o^2(t_r)} = \frac{1}{2}\frac{kT}{C_D}\left(1 - \frac{\delta^2}{(t_r - t_1 + \delta)^2}\right),$$

  where

  $t_1 < 1$ns is the time to reach subthreshold and

  $\delta = \frac{v_t C_D}{i_d(t_1)} \approx 6$ns

- Since $t_r >> \delta$, the average output noise simplifies to

$$\overline{V_o^2(t_r)} \approx \frac{1}{2}\frac{kT}{C_D},$$

  which is half its steady state value !!

- Note that reset noise in general depends on photo and dark current. In practice, however, $t_r >> \delta$ and reset noise is independent of photocurrent, dark current, and integration time (same as for PPS)

- Even though soft reset results in lower noise than hard reset, it can cause image lag.

  This is because the final reset voltage depends on the signal value. After a dark frame, the reset begin in subthreshold, while after a bright frame it begins in ohmic region

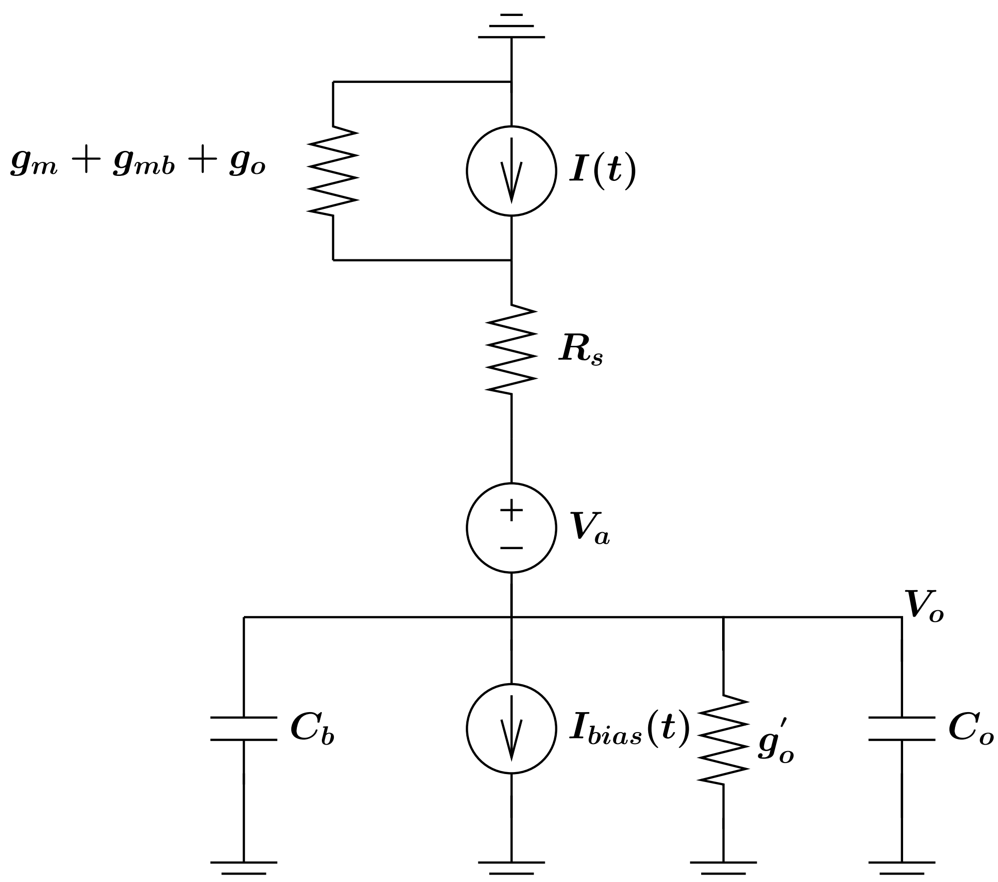
  This problem can be solved by combining the two schemes:

  1. Set the the reset transistor drain at a low voltage ($< v_{DD} - v_{TR}$)
  2. Perform "hard reset" to that voltage by setting the reset gate to v_{DD}
  3. Bring the the reset transistor drain to v_{DD} and perform soft reset

  Most of the reset time is spent on the soft reset phase

Ouput Noise Due to Readout

- We use the MOS transistor small signal model



- The noise sources are all independent and zero mean with psds

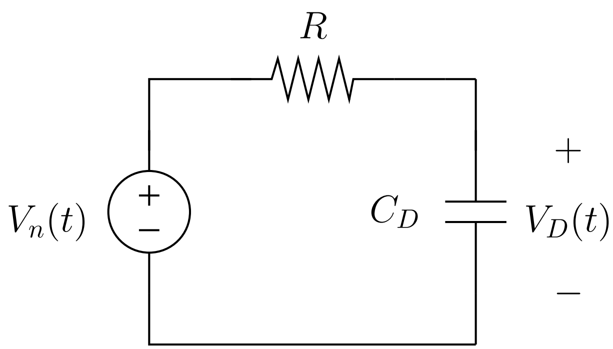
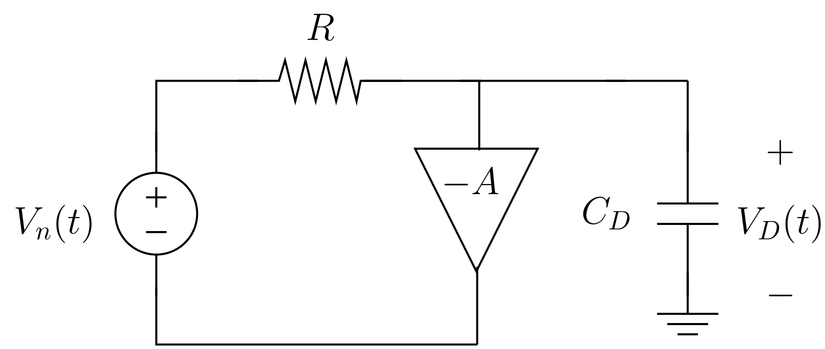
$$S_I(f) = 2kT\left(\frac{2}{3}g_m\right)$$
$$S_{V_a}(f) = 2kTR_s$$
$$S_{I_{bias}}(f) = 2kT\left(\frac{2}{3}g_m'\right)$$

- We can assume that the circuit is time invariant and in steady state (long enough readout time) and analyze the noise for each source separartely and then add up their contributions (which you will do in the homework)

- Noise due to readout is independent of photocurrent, dark current, and integration time

Active Reset

- Reset noise in 3T APS can be reduced by placing the reset transistor in the feedback loop of a gain stage

Hard Reset	Active Reset
	
$\frac{V_D(f)}{V_n(f)} = \frac{1}{1+j2\pi f RC_D}$	$\frac{V_D(f)}{V_n(f)} \approx \frac{1/A}{1+j2\pi f\left(\frac{RC_D}{A}\right)}$
$\overline{V_D^2(t)} = \frac{kT}{C_D}$	$\overline{V_D^2(t)} = \frac{kT}{AC_D}$

- The gain stage increases the bandwidth by A, but reduces the magnitude by A. As a result, the noise power goes down by A

- Now, consider the more realistic case where the gain stage has a limited bandwidth f_o, i.e., replace A by

$$A(f) = \frac{A}{1 + j\frac{f}{f_o}}$$

In this case the transfer function becomes

$$\begin{aligned}
\frac{V_D(f)}{V_n(f)} &= \frac{1 + j\frac{f}{f_o}}{\left(1 + j\frac{f}{f_r}\right)\left(1 + j\frac{f}{f_o}\right) + A} \\
&= \frac{1}{A}\left(\frac{1 + j\frac{f}{f_o}}{1 - f^2\frac{1}{Af_of_r} + jf\frac{(f_o+f_r)}{Af_of_r}}\right),
\end{aligned}$$

where $f_r = 1/2\pi C_D R$

The modulus square of the transfer function is given by

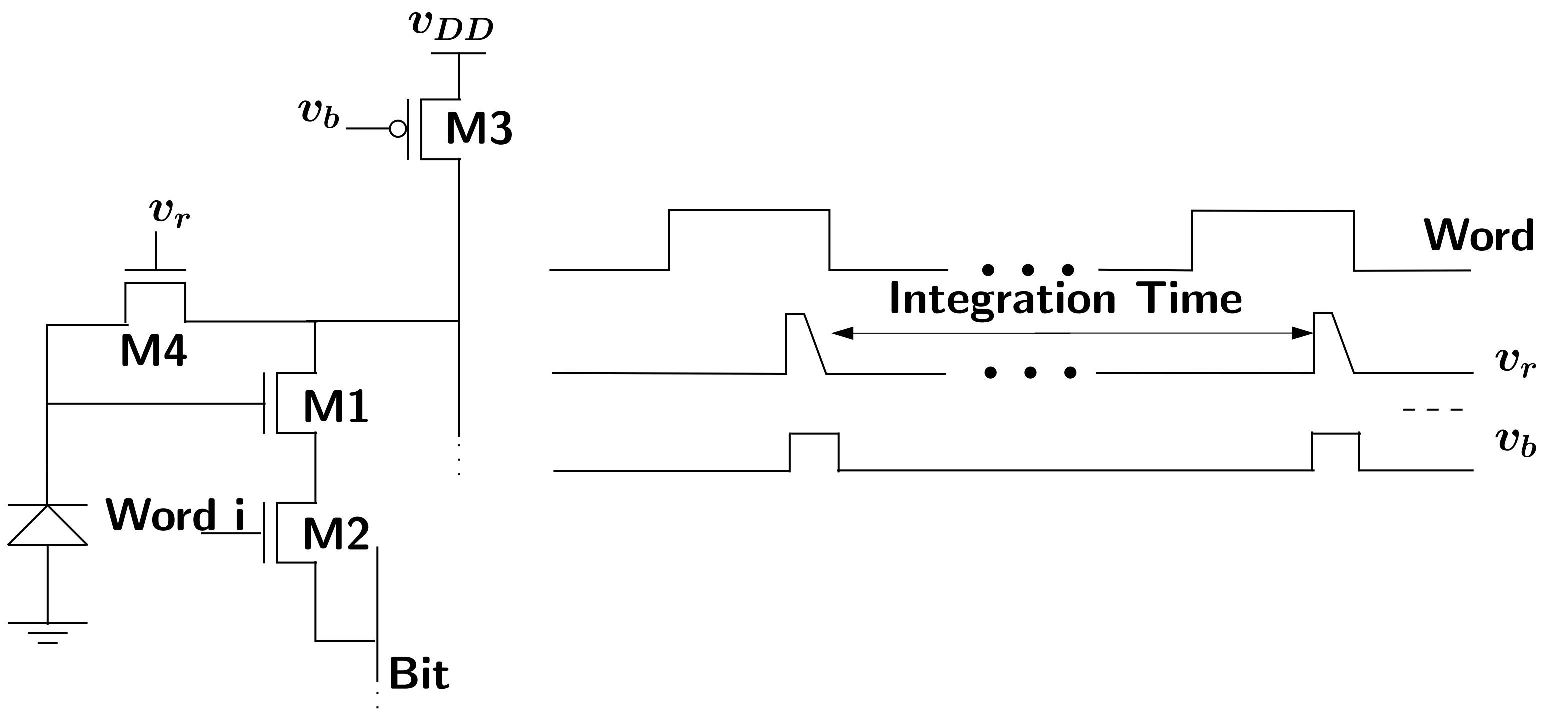
$$\left|\frac{V_D(f)}{V_n(f)}\right|^2 = \frac{1}{A^2}\left(\frac{1 + \left(\frac{f}{f_o}\right)^2}{\left(1 - \frac{f^2}{Af_of_r}\right)^2 + \left(\frac{f(f_o+f_r)}{Af_of_r}\right)^2}\right)$$

Now, we want the denominator to be as large as possible, so we set f_r as small as possible, i.e.,

$$R \gg \frac{1}{2\pi C_D}$$

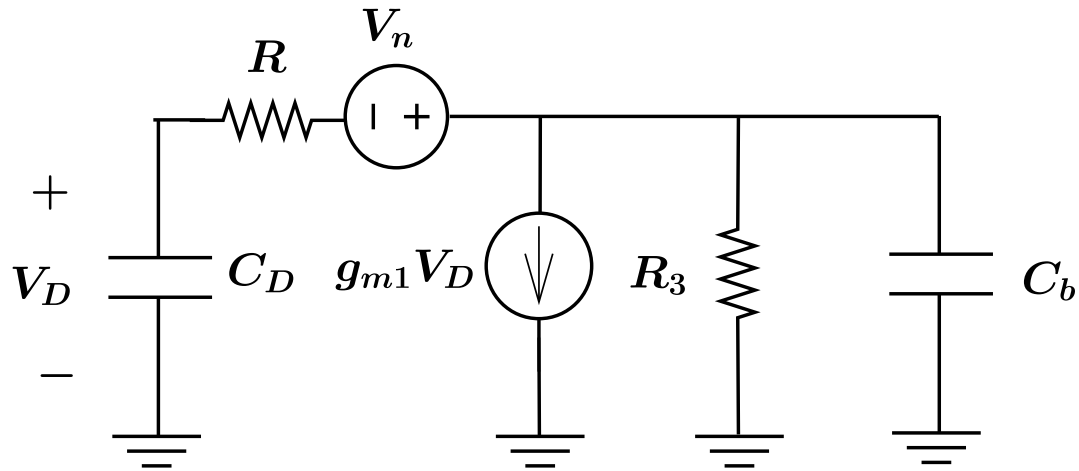
So, the resistance of the reset transistor must be very high

- Column level implementation [Kozlowski et al.'05]:



- During readout M2 is on, M3 is on, and M4 is off
- During reset:
  - M3 acts as a current source (biased by v_b)
  - M1 (together with M3) act as a common source amplifier
  - M2 is on
  - v_r is tapered low such that M4 has very high resistance, R

- To obtain more accurate results, we include the bitline capacitance C_b and the resistance of M3, R_3, in the small signal circuit



The transfer function is

$$\frac{V_D(f)}{V_n(f)} = -\frac{1 + j2\pi f C_b R_3}{1 + g_{m1}R_3 + j2\pi f(C_D R + C_b R_3 + C_D R_3) - 4\pi^2 f^2 C_D C_b R_3 R}$$

Here the gain is $g_{m1}R_3$ and $f_o = 1/2\pi C_b R_3$

Assuming the gain is 100 and $R = 10^9$ Ohm, we obtain

$$\overline{V_D^2} = 5.86 \times 10^{-8}\ \mathrm{V}^2,$$

which is 3.53 times lower than kT/C_D.

Appendix: Noise Analysis Background

- Random Variables
- Random Processes
- Analysis of Noise in Circuits

Random Variables

- Noise samples are modeled as random variables (r.v.s)
- A discrete r.v. $X \in \mathcal{X}$, where $\mathcal{X}$ is a discrete set of real numbers, is completely specified by its *probability mass functions* (pmf):

$$p(x) \geq 0, \quad \sum_{x \in \mathcal{X}} p(x) = 1$$

  Example: Poisson r.v. with rate $\lambda > 0$ $X \sim \mathrm{Poisson}(\lambda)$:

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \text{ for } k = 0, 1, 2, \ldots$$

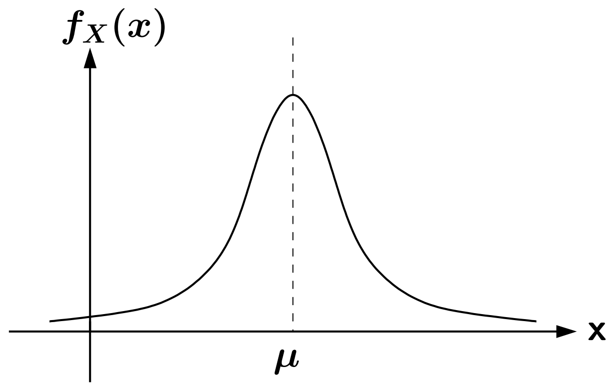
  Models the number of random and independent events in a unit time, e.g., number of photons arriving at the surface of an image sensor per unit time, or the number of electrons generated in a semiconductor region per unit time

- A continuous r.v. $X \in (-\infty, \infty)$ is completely specified by its *probability density function* (pdf):

$$f_X(x) \geq 0, \quad \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

Example: Gaussian r.v. ($X \sim \mathcal{N}(\mu, \sigma^2)$):

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



- First and second moments:
  - *Mean* of X: $\overline{X} = \int_{-\infty}^{\infty} x f(x)\, dx$
  - *Mean square* of X, i.e., average power: $\overline{X^2} = \int_{-\infty}^{\infty} x^2 f(x)\, dx$
  - *Root mean square* (rms) of X: rms(X)= $\sqrt{\overline{X^2}}$
  - *Variance* of X: $\sigma_X^2 = \overline{X^2} - \overline{X}^2$ (it measures the mean square distance of X from its mean $\overline{X}$)

- Two continuous random variables X and Y are specified by the joint pdf

$$f(x,y) \geq 0, \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

  - X and Y are *independent* if $f(x,y) = f_X(x)f_Y(y)$ for all (x,y)
  - The *correlation* $\overline{X \cdot Y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy$ (it measures linear independence)
  - X,Y are *uncorrelated* if $\overline{X \cdot Y} = \overline{X} \cdot \overline{Y}$ (in this case they are linearly independent)
  - If X and Y are independent then they are uncorrelated – converse does not hold in general

Random Processes

- Noise waveforms are modeled as random processes
- A random process $X(t)$, $-\infty < t < \infty$, is an infinite collection of random variables (indexed by time t)
- For any time instances t_1, $t_2, \ldots$, t_n, the samples $X(t_1)$, $X(t_2), \ldots$, $X(t_n)$ are random variables
  - Process *mean*: $\overline{X}(t)$
  - Process *autocorrelation*: $R_X(t+\tau, t) = \overline{X(t+\tau) \cdot X(t)}$
- Many important noise sources are modeled as *stationary* random processes, i.e., processes with time invariant statistics
- If $X(t)$ is a stationary random process then its mean and autocorrelation functions are time invariant, i.e.,

$$\begin{aligned} \overline{X}(t) &= \mu \\ R_X(t+\tau, t) &= R_X(\tau) \end{aligned}$$

- For a stationary random process
  - $R_X(0) = \overline{X^2(t)}$, i.e., mean square or average power of $X(t)$
  - $R_X(\tau)$ is an even function, and has maximum at $\tau = 0$, i.e., $|R_X(\tau)| \leq R_X(0)$ for all τ
- The Fourier Transform of $R_X(\tau)$

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau}\, d\tau, \quad -\infty < f < \infty$$

is called the *power spectral density* (psd) of the stationary process $X(t)$

  - $S_X(f) \geq 0$ and is even
  - The average power $\overline{X^2(t)} = \int_{-\infty}^{\infty} S_X(f)\, df$
  - The average power in any frequency band $[f_1, f_2]$ is

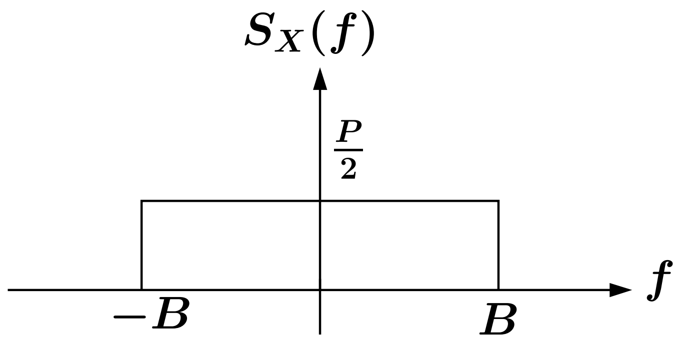
$$2\int_{f_1}^{f_2} S_X(f)\, df$$

  - Note: The psd is often defined as a one sided function, which (for a zero mean process) is equal to $2S_X(f)$, for $f \geq 0$, and zero, for $f < 0$

White Noise

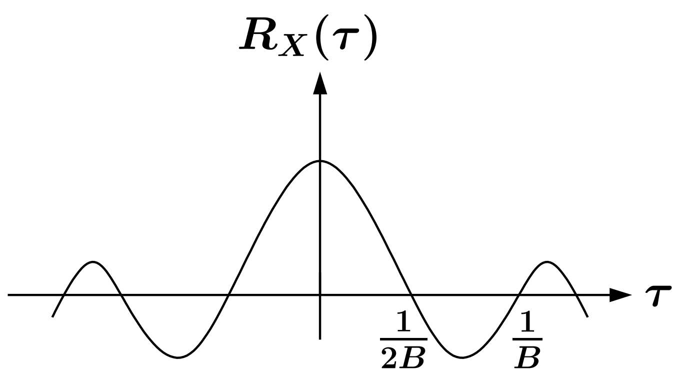
- A *bandlimited white noise* process is a stationary process with $\overline{X}(t) = 0$, and

$$S_X(f) = \begin{cases} \frac{P}{2} & |f| < B \\ 0 & \text{otherwise} \end{cases}$$



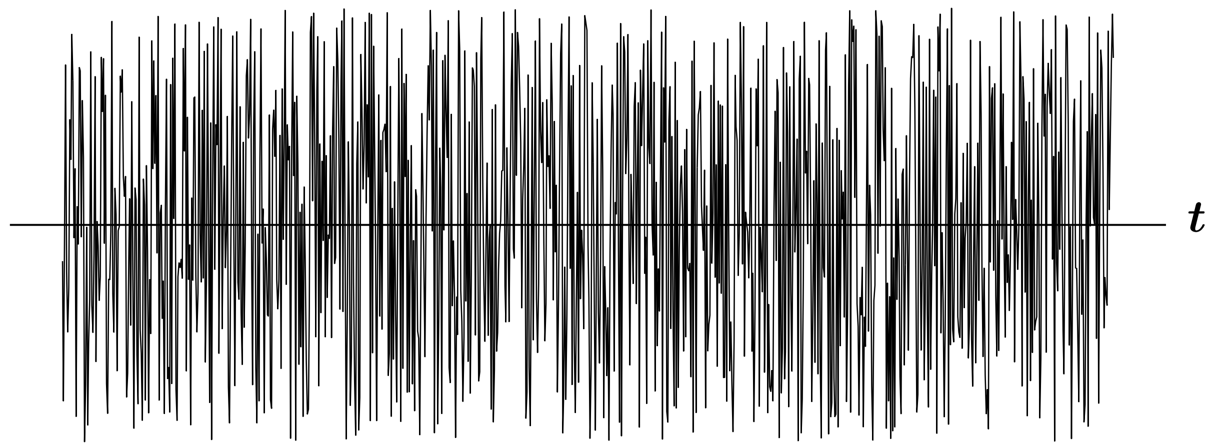
Thus its autocorrelation function

$$R_X(\tau) = PB\frac{\sin 2\pi B\tau}{2\pi B\tau} = PB \operatorname{sinc} 2B\tau$$



And, the average power $\overline{X^2}(t) = \int_{-\infty}^{\infty} S_X(f)\, df = PB$

- If we let $B \to \infty$, we get a *white noise* process, i.e., a stationary process with $\overline{X}(t) = 0$, $R_X(\tau) = \frac{P}{2}\delta(\tau)$ (all samples from the process are uncorrelated), and $S_X(f) = \frac{P}{2}$ for all f



- White noise process is not physically realizable, since it has infinite power, but is very useful in modeling important sources of noise, e.g., *thermal* and *shot* noise

- If $X(t)$ (for any t) is a Gaussian r.v., then the white noise process is called *white Gaussian noise* (WGN) (thermal and shot noise are WGN)

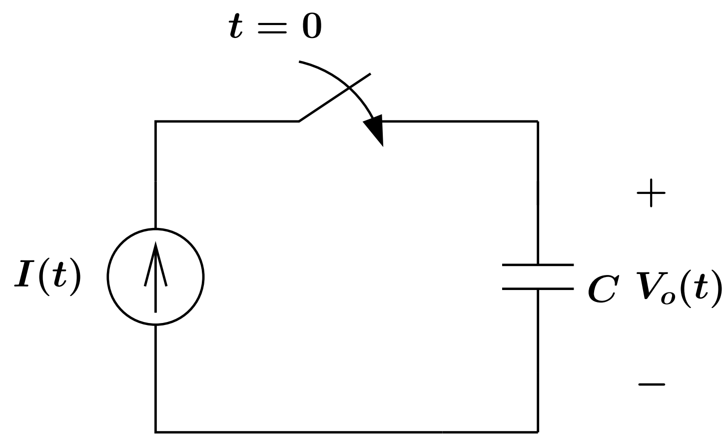
- A Poisson point process models random and independent events in time, e.g., arrival of photons at the surface of an image sensor, carrier generation in a semiconductor, current across a depletion region, etc
  - The number of arrivals per unit time is Poisson
  - Important fact: When the rate is sufficiently high, the process after subtracting the mean approaches a WGN process

Analysis of noise in circuits

- Since noise signals are typically very small, we can use small signal circuit models (around the operating voltages and currents) to analyze their effects on the circuit output(s)
- In general (even though we use linear circuit models) the analysis can be complicated if the circuit is not in steady state, if the circuit parameters are time varying, or if the noise is not stationary
- We discuss two important special cases that are extensively used in the analysis of noise in circuits
  - When the circuit is an integrator (capacitor)
  - When the circuit is time invariant and in steady state

  In both cases, we assume that the noise is stationary
- Remark: these two cases are not sufficient for performing complete and accurate analysis of noise in image sensors (as we will see)

- If the circuit is a capacitor and we apply a zero mean stationary current source process $I(t)$ to it at time $t = 0$



Then for $t > 0$ the output noise voltage

$$V_o(t) = \frac{1}{C}\int_0^t I(\tau)\, d\tau$$

Thus $\overline{V_o(t)} = 0$ and the average power

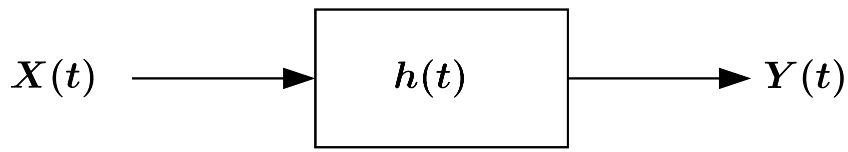
$$\begin{aligned}\overline{V_o^2(t)} &= \frac{1}{C^2}\int_0^t \int_0^t \overline{I(t_1)\cdot I(t_2)}\, dt_1\, dt_2 \\ &= \frac{1}{C^2}\int_0^t \int_0^t R_I(t_1 - t_2)\, dt_1\, dt_2 = \frac{1}{C^2}\int_{-t}^t (t - |\tau|) R_I(\tau)\, d\tau\end{aligned}$$

where $\tau = t_1 - t_2$, and $R_I(\tau)$ is the process autocorrelation function

If $I(t)$ is a white noise process with psd $P/2$, then $R_I(\tau) = \frac{P}{2}\delta(\tau)$ and

$$\begin{aligned}\overline{V_o^2(t)} &= \frac{1}{C^2}\int_{-t}^{t}(t-|\tau|)\frac{P}{2}\delta(\tau)d\tau \\ &= \frac{P}{2C^2}t\end{aligned}$$

- If the circuit is *linear time invariant* with impulse response $h(t)$ and input zero mean WSS (voltage or current source) process $X(t)$



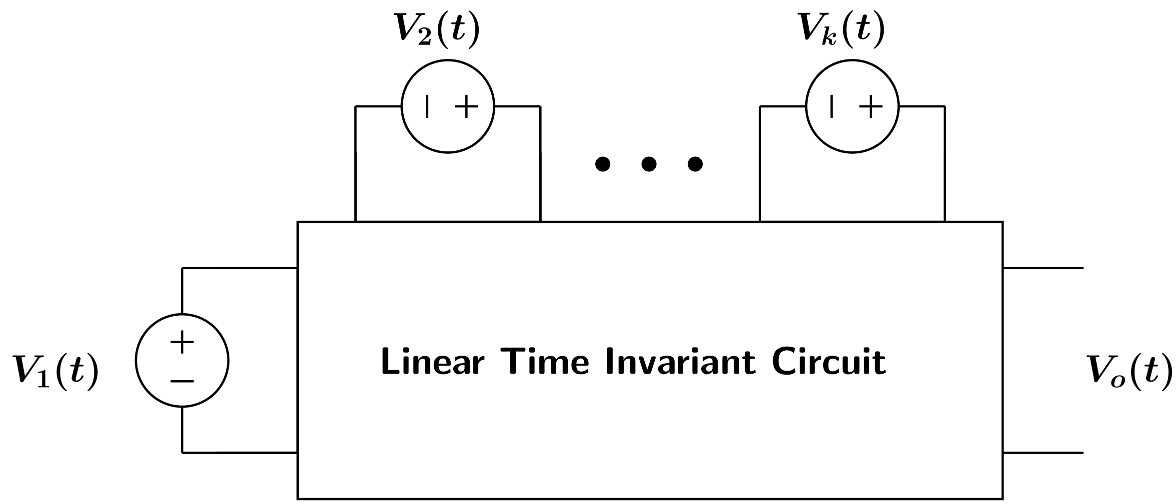
then in steady state, the output process $Y(t)$ is also WSS with zero mean and

$$S_Y(f) = |H(f)|^2 S_X(f)$$

where $H(f) = \mathcal{F}[h(t)]$ is the circuit transfer function

Superposition for Uncorrelated Sources

- Consider a linear time invariant circuit with multiple zero mean uncorrelated WSS noise sources $V_1(t)$, $V_2(t), \ldots$, $V_k(t)$



- In steady state the output noise $V_o(t)$ is a zero mean WSS with psd

$$S_{V_o}(f) = \sum_{i=1}^{k} |H_i(f)|^2 S_{V_i}(f),$$

where $H_i(f)$ is the *transfer function* from the ith source to the output

- Thus, the average output power is

$$\overline{V_o^2(t)} = \int_{-\infty}^{\infty} S_{V_o}(f)\, df = \sum_{i=1}^{k} \overline{V_{oi}^2(t)}$$

- So, we can obtain the average noise power by superposition:
  - Construct the small signal model of the circuit around its bias
  - Turn on one source at a time, find the transfer function from the source to the output, and find its contribution to the average output noise power
  - Add up the contributions from all sources
- Input referred noise:
  - If the transfer function of the system is linear, then both the output noise power (which is computed using the small signal model) and the output signal power are scaled by the same gain factor. Thus SNR can be computed either at the input or output
  - In general the noise power must be first referred to the input using the small signal gain of the circuit before computing SNR