Lecture #13  Joint Source–Channel Coding

(Reading: NIT 3.9, 14.1)

- Communication of a DMS over a DMC
- Communication of a 2-DMS over a DM-MAC
  - A joint source–channel coding scheme
  - Common part of a 2-DMS
  - A more general joint source–channel coding scheme

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Communication of a DMS over a DMC

\[ U^k \xrightarrow{\text{Encoder}} X^n \xrightarrow{\text{Channel}} Y^n \xrightarrow{\text{Decoder}} \hat{U}^k \]

- DMS \( U \) with entropy \( H(U) \)
- DMC \( p(y|x) \) with capacity \( C \)
- A \( (|U|^k, n) \) joint source–channel code of rate \( r = k/n \) symbols/transmission:
  - Encoder: \( x^n(u^k) \in X^n \)
  - Decoder: \( \hat{x}^k(y^n) \in \hat{U}^k \)
- \( r \) achievable if \( \exists (|U|^{rn}, n) \) codes such that \( \lim_{n \to \infty} P\{\hat{U}^{rn} \neq U^{rn}\} = 0 \)
- What is the necessary and sufficient condition for \( r \) to be achievable?
Joint source–channel coding

Source–channel separation theorem (Shannon 1959)

- If \( rH(U) < C \), then \( r \) is achievable
- If \( r \) is achievable, then \( rH(U) \leq C \)

- Proof of the converse:

\[
H(U) \leq \frac{1}{k} I(U^k; \hat{U}^k) \\
\leq \frac{1}{k} I(U^k; Y^n) \\
\leq \frac{1}{k} \sum_{i=1}^{n} I(X_i; Y_i) \leq \frac{1}{r} C
\]

Proof of achievability

- Separate source and channel coding is asymptotically optimal
- \( P\{\hat{U}^n \neq U^n\} \leq P\{\hat{U}^n \neq U^n, \hat{M} = M\} + P\{\hat{M} \neq M\} \)
- **Source coding**: \( P\{\hat{U}^n \neq U^n, \hat{M} = M\} \to 0 \) if \( R > H(U) \) bits/symbol
- **Channel coding**: \( P\{\hat{M} \neq M\} \to 0 \) if \( rR < C \)
- Combined together, \( P\{\hat{U}^n \neq U^n\} \to 0 \) if \( rH(U) < C \)
- **Basis for digital communication**: Bits as “universal” source–channel interface
Communication of a 2-DMS over a DM-MAC

- 2-DMS $(U_1, U_2)$ and a DM-MAC $p(y|x_1, x_2)$
- A $(|U_1|^{k_1}, |U_2|^{k_2}, n)$ joint source–channel code of rate pair $(r_1 = k_1/n, r_2 = k_2/n)$:
  - Encoder $j = 1, 2$: $x_j^n(u_j^{k_j})$
  - Decoder: $(\hat{U}_1^{k_1}(y^n), \hat{U}_2^{k_2}(y^n))$
- Probability of error: $P_e^{(n)} = P\{(\hat{U}_1^{k_1}, \hat{U}_2^{k_2}) \neq (U_1^{k_1}, U_2^{k_2})\}$
- Lossless communication if $\exists (|U_1|^{k_1}, |U_2|^{k_2}, n)$ codes with $\lim_{n\to\infty} P_e^{(n)} = 0$
- What is the necessary and sufficient condition for lossless communication?
- For simplicity, assume that $r_1 = r_2 = 1$ symbol/transmission

A separate source and channel coding scheme

- $\mathcal{C}$ of the DM-MAC is the set of $(R_1, R_2)$ such that
  \[
  R_1 \leq I(X_1; Y|X_2, Q), \\
  R_2 \leq I(X_2; Y|X_1, Q), \\
  R_1 + R_2 \leq I(X_1, X_2; Y|Q)
  \]
  for some $p(q)p(x_1|q)p(x_2|q)$
- $\mathcal{R}^*$ for distributed lossless compression is the set of $(R_1, R_2)$ such that
  \[
  R_1 \geq H(U_1|U_2), \\
  R_2 \geq H(U_2|U_1), \\
  R_1 + R_2 \geq H(U_1, U_2)
  \]
  Hence, separation achieves lossless communication if $\mathcal{C} \cap \mathcal{R}^*$ is not empty:
  \[
  H(U_1|U_2) < I(X_1; Y|X_2, Q), \\
  H(U_2|U_1) < I(X_2; Y|X_1, Q), \\
  H(U_1, U_2) < I(X_1, X_2; Y|Q)
  \]
  for some $p(q)p(x_1|q)p(x_2|q)$
Examples

- **MAC with orthogonal components** \( p(y|x_1, x_2) = p(y_1|x_1)p(y_2|x_2) \):
  Lossless communication is possible iff
  \[
  H(U_1|U_2) \leq C_1, \\
  H(U_2|U_1) \leq C_2, \\
  H(U_1, U_2) \leq C_1 + C_2
  \]

- **Independent sources** \((U_1, U_2) \sim p(u_1)p(u_2)\):
  Lossless communication is possible iff
  \[
  H(U_1) \leq -I(X_1; Y|X_2, Q), \\
  H(U_2) \leq -I(X_2; Y|X_1, Q), \\
  H(U_1) + H(U_2) \leq -I(X_1, X_2; Y|Q)
  \]
  for some \(p(q)p(x_1|q)p(x_2|q)\)

Counterexample (Cover–El Gamal–Salehi 1980)

- Let \((U_1, U_2)\) be a 2-DMS with \(p_{U_1, U_2}(0, 0) = p_{U_1, U_2}(0, 1) = p_{U_1, U_2}(1, 1) = 1/3\)
- Send \((U_1, U_2)\) over a binary erasure MAC \((X_1, X_2 \in \{0, 1\}, \ Y = X_1 + X_2)\)
- \(\mathcal{R}^* \cap \mathcal{C} = \emptyset\)

- Cannot send \((U_1, U_2)\) losslessly using this separate source and channel scheme
- Now consider uncoded transmission: \(X_{1i} = U_{1i}, X_{2i} = U_{2i}, i \in [1 : n]\)
- Hence, joint source–channel coding achieves error-free transmission!
- Separation does not hold in general for sending sources over multiuser channels
A joint source–channel coding scheme

The 2-DMS \((U_1, U_2)\) can be sent losslessly over a DM-MAC \(p(y|x_1, x_2)\) if

\[
H(U_1 | U_2) < I(X_1; Y | X_2, U_2, Q),
\]
\[
H(U_2 | U_1) < I(X_2; Y | X_1, U_1, Q),
\]
\[
H(U_1, U_2) < I(X_1, X_2; Y | Q)
\]

for some \(p(q, x_1, x_2 | u_1, u_2) = p(q)p(x_1 | u_1, q)p(x_2 | u_2, q)\)

- Special cases:
  - **Separate source and channel coding**: Set \(p(x_1 | u_1, q)p(x_2 | u_2, q) = p(x_1 | q)p(x_2 | q)\)
  - **Counterexample**: Set \(Q = \emptyset, X_1 = U_1,\) and \(X_2 = U_2\)

Proof of achievability

- Assume \(|Q| = 1\) (the general case follows by coded time sharing)
- **Codebook generation**: Fix \(p(x_1 | u_1)\) and \(p(x_2 | u_2)\)
  - For each \(u_1^n \in U_1^n\), randomly and independently generate \(x_1^n(u_1^n) \sim \prod_{i=1}^{n} p_{X_1 | U_1}(x_1_i | u_1_i)\)
  - For each \(u_2^n \in U_2^n\), randomly and independently generate \(x_2^n(u_2^n) \sim \prod_{i=1}^{n} p_{X_2 | U_2}(x_2_i | u_2_i)\)
- **Encoding**:
  - Upon observing \(u_1^n\), encoder 1 transmits \(x_1^n(u_1^n)\)
  - Upon observing \(u_2^n\), encoder 2 transmits \(x_2^n(u_2^n)\)
  - No more than \(2^{n(H(U_1, U_2) + \delta(\epsilon))}\) codeword pairs \((x_1^n, x_2^n)\) can simultaneously occur w.h.p.
- **Decoding**:
  - Find the unique pair \((\hat{u}_1^n, \hat{u}_2^n)\) such that \((\hat{u}_1^n, \hat{u}_2^n, x_1^n(\hat{u}_1^n), x_2^n(\hat{u}_2^n), y^n) \in T_\epsilon^{(n)}\)
Analysis of the probability of error

- Consider the error events:
  \[ E_1 = \{ (U_1^n, U_2^n, X_1^n(U_1^n), X_2^n(U_2^n), Y^n) \notin T^{(n)}_\epsilon \} \]
  \[ E_2 = \{ (\bar{u}_1^n, U_2^n, X_1^n(\bar{u}_1^n), X_2^n(U_2^n), Y^n) \in T^{(n)}_\epsilon \text{ for some } \bar{u}_1^n \neq U_1^n \} \]
  \[ E_3 = \{ (U_1^n, \bar{u}_2^n, X_1^n(U_1^n), X_2^n(\bar{u}_2^n), Y^n) \in T^{(n)}_\epsilon \text{ for some } \bar{u}_2^n \neq U_2^n \} \]
  \[ E_4 = \{ (\bar{u}_1^n, \bar{u}_2^n, X_1^n(\bar{u}_1^n), X_2^n(\bar{u}_2^n), Y^n) \in T^{(n)}_\epsilon \text{ for some } \bar{u}_1^n \neq U_1^n, \bar{u}_2^n \neq U_2^n \} \]

- Then, the average probability of error
  \[ P(E) \leq P(E_1) + P(E_2) + P(E_3) + P(E_4) \]

- Cannot use the packing lemma or joint typicality lemma to bound \( P(E_j) \), \( j = 2, 3, 4 \)

- Use basic properties of joint typicality (see NIT 14.1.2)

Suboptimality of the scheme

- Let \( U_1 = U_2 = U \)
- The scheme yields:
  \[ H(U) < \max_{p(x_1|u)p(x_2|u)} I(X_1, X_2; Y) \]
  Cannot generate all joint pmfs on \( (X_1, X_2) \) in general

- But, since both senders observe same source, can use cooperative coding:
  \[ H(U) < \max_{p(x_1, x_2)} I(X_1, X_2; Y), \]
  which can be less stringent than using the joint source-channel coding scheme

- In general the scheme can be improved when \( (U_1, U_2) \) have a common part
Common part of a 2-DMS

- Let \((U_1, U_2) \sim p(u_1, u_2)\). Arrange \(p(u_1, u_2)\) in largest block diagonal form:

\[
\begin{array}{ccc}
  u_0 = 1 & 0 & \cdots & 0 \\
  0 & u_0 = 2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & u_0 = k \\
\end{array}
\]

- Common part between \(U_1\) and \(U_2\): \(U_0 = u_0\) if \((U_1, U_2)\) is in block \(u_0 \in [1 : k]\).

- Formally (Gács–Körner 1973, Witsenhausen 1975):
  Let \(g_j : U_j \rightarrow [1 : k], j = 1, 2,\) be functions with largest \(k\) such that:
  \[
P(g_j(u_j) = u_0) > 0 \text{ for every } u_0 \in [1 : k], j = 1, 2, \text{ and } P(g_1(U_1) = g_2(U_2)) = 1
  \]
  Then the common part is \(U_0 = g_1(U_1) = g_2(U_2)\).

Common part of a 2-DMS \((U_1, U_2)\)

- Example:

\[
\begin{array}{cccc}
  u_0 = 1 & 0.1 & 0.2 & 0 & 0 \\
  u_0 = 2 & 0.1 & 0.1 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  u_0 = 4 & 0 & 0 & 0.2 & 0.1 \\
\end{array}
\]

Here \(k = 2, P(U_0 = 1) = 0.7, P(U_0 = 2) = 0.3\).

- Now, consider a 2-DMS \((U_1, U_2)\).

- Common part between \(U_1^n\) and \(U_2^n\) is \(U_0^n\) (up to relabeling).

- Hence, the DMS \(U_0\) is the common part of the 2-DMS \((U_1, U_2)\).

- Can modify our joint source–channel scheme to include common part.
Three-index separate source and channel coding scheme

- Source coding:
  
  \[ U^n \xrightarrow{\text{Encoder 1}} M_1 \]
  \[ U^n \xrightarrow{\text{Encoder 2}} M_2 \]

  - \( M_0 \) can depend only on \( U^n_0 \)
  - Hence, the optimal rate region \( R^* \) is the set of \((R_0, R_1, R_2)\) such that
    
    \[
    R_1 \geq H(U_1 | U_2), \\
    R_2 \geq H(U_2 | U_1), \\
    R_1 + R_2 \geq H(U_1, U_2 | U_0), \\
    R_0 + R_1 + R_2 \geq H(U_1, U_2)
    \]

  - Lossless compression of \( U_0 \); S–W coding of \((U_1, U_2)\) for every \( u^n_0 \)

Three-index separate source and channel coding scheme

- Channel coding:

  \[ M_0, M_1 \xrightarrow{\text{Encoder 1}} X^n_1 \]
  \[ M_0, M_2 \xrightarrow{\text{Encoder 2}} X^n_2 \]

  \[ p(y|x_1, x_2) \]
  \[ Y^n \xrightarrow{\text{Decoder}} \hat{M}_0, \hat{M}_1, \hat{M}_2 \]

  - The capacity region \( C \) with a common message is the set of \((R_0, R_1, R_2)\) such that
    
    \[
    R_1 \leq I(X_1; Y | X_2, W), \\
    R_2 \leq I(X_2; Y | X_1, W), \\
    R_1 + R_2 \leq I(X_1, X_2; Y | W), \\
    R_0 + R_1 + R_2 \leq I(X_1, X_2; Y)
    \]

    for some \( p(w)p(x_1|w)p(x_2|w) \) with \(|\mathcal{W}| \leq \min\{|\mathcal{X}_1| \cdot |\mathcal{X}_2| + 2, |\mathcal{Y}| + 3\} \) (Problem 5.19)

  - Use superposition coding
Three-index separate source and channel coding scheme

- Hence, separation achieves lossless communication if $C \cap R^*$ is not empty:

$$H(U_1 | U_2) < I(X_1; Y | X_2, W),$$
$$H(U_2 | U_1) < I(X_2; Y | X_1, W),$$
$$H(U_1, U_2 | U_0) < I(X_1, X_2; Y | W),$$
$$H(U_1, U_2) < I(X_1, X_2; Y)$$

for some $p(w)p(x_1|w)p(x_2|w)$ with $|W| \leq \min\{|X_1| \cdot |X_2| + 2, |Y| + 3\}$

A joint source–channel coding scheme with common part

- Combine three-index coding scheme and Theorem 14.1:


A 2-DMS with common part can be sent losslessly over a DM-MAC if

$$H(U_1 | U_2) < I(X_1; Y | X_2, U_2, W),$$
$$H(U_2 | U_1) < I(X_2; Y | X_1, U_1, W),$$
$$H(U_1, U_2 | U_0) < I(X_1, X_2; Y | U_0, W),$$
$$H(U_1, U_2) < I(X_1, X_2; Y)$$

for some $p(w)p(x_1|u_1, w)p(x_2|u_2, w)$

- $U_0$ is represented by $W$, chosen to maximize cooperation between the senders
- Use the previous joint source–channel scheme for each $w''$
- Optimal for the DM-MAC with common message (Slepian–Wolf 1973)
- Not optimal in general (Dueck 1981)
Summary

- Source channel separation holds for point to point communication
- Source–channel separation does not hold in general for multiuser channels
- Joint source–channel coding schemes that utilize the correlation between the sources for cooperative transmission
- Common part of a 2-DMS

References


