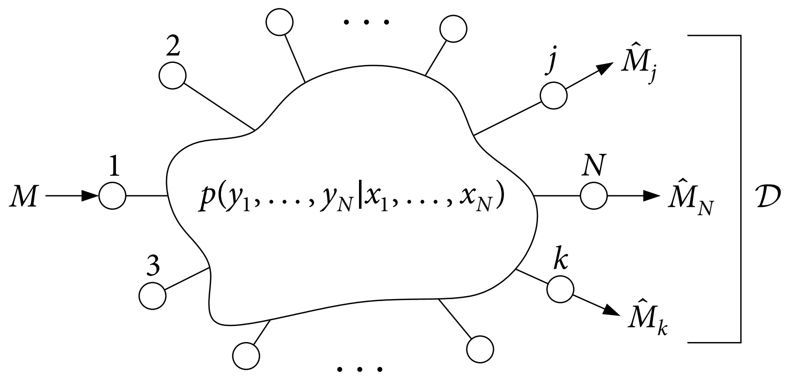
Lecture #12 General Multihop Networks

(Reading: NIT 18.1–18.3, 19.1)

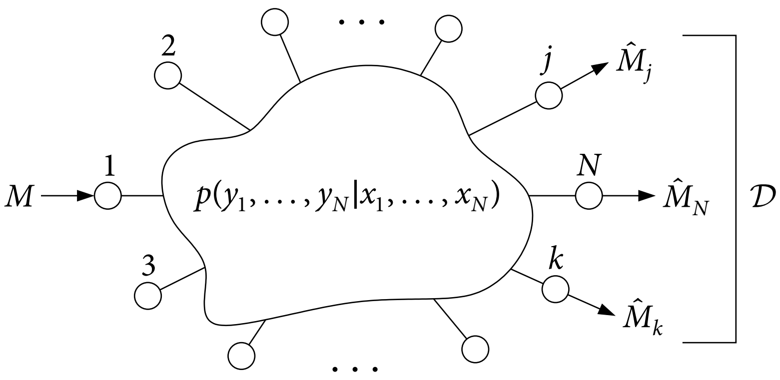
- DM multicast network
- Network decode–forward
- Noisy network coding
- Gaussian network

Discrete memoryless multicast network



- N-node DM multicast network (DM-MN) $(\times_{j=1}^N \mathcal{X}_j, p(y^N|x^N), \times_{j=1}^N \mathcal{Y}_j)$
- A $(2^{nR}, n)$ code:
  - Message set: $[1:2^{nR}]$
  - Source encoder: $x_{1i}(m, y_1^{i-1})$, $i \in [1:n]$
  - Relay encoder $j \in [2:N]$: $x_{ji}(y_j^{i-1})$, $i \in [1:n]$
  - Decoder $k \in \mathcal{D}$: $\hat{m}_k(y_k^n)$
- $P_e^{(n)}$, achievability, and the capacity C defined as before

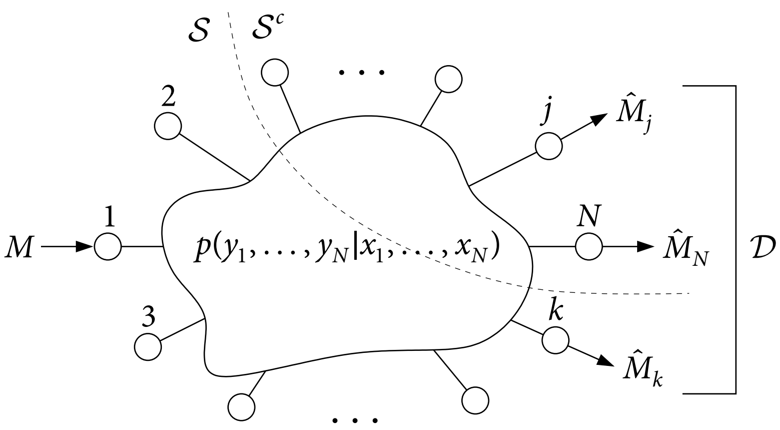
Discrete memoryless multicast network



- Special case:
  - Graphical multicast networks: DMN decomposes into orthogonal DMCs
  - DM-RC: $N = 3$, $X_3 = Y_1 = \emptyset$, and $\mathcal{D} = \{3\}$
  - DMC with feedback: $N = 2$, $Y_1 = Y_2$, $X_2 = \emptyset$, and $\mathcal{D} = \{2\}$
  - DM unicast network: $\mathcal{D} = \{N\}$
  - Common-message DM-BC: $X_2 = \cdots = X_N = Y_1 = \emptyset$ and $\mathcal{D} = [2:N]$
- Capacity of the DM-MN is not known in general (e.g., DM-RC)
- We discuss upper and lower bounds that are tight in some cases

Cutset upper bound

- Cutset bounds for graphical networks and the RC can be generalized to DM-MN



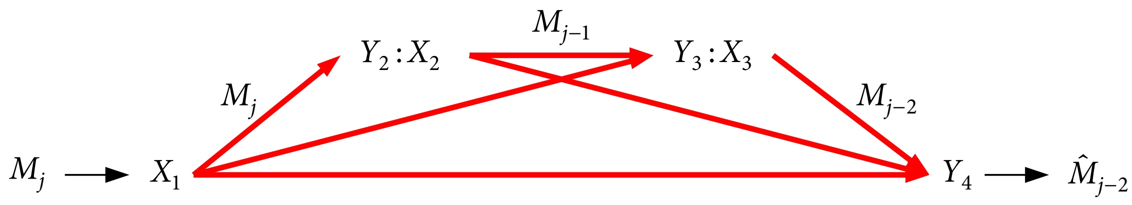
Theorem 18.1 (Cutset upper bound) (El Gamal 1981)

$$C \le \max_{p(x^N)} \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, k \in \mathcal{S}^c} I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c)),$$

where $X(\mathcal{S}) = \{X_j : j \in \mathcal{S}\}, \dots$

Network decode–forward

- Decode–forward for RC can be extended to DM-MN



Theorem 18.2 (Xie–Kumar 2005, Kramer–Gastpar–Gupta 2005)

$$C \geq \max_{p(x^N)} \min_{k \in [1:N-1]} I(X^k; Y_{k+1} | X_{k+1}^N)$$

- Tight for degraded DM-MN, $p(y_{k+2}^N | x^N, y^{k+1}) = p(y_{k+2}^N | x_{k+1}^N, y_{k+1})$, $k \in [1 : N-2]$
- Holds for any $\mathcal{D} \subseteq [2 : N]$
- Can be improved by removing some relay nodes and relabeling nodes

Proof of achievability (for DM-RC)

- Key idea: sliding window decoding (Carleial 1982)
- Codebook generation, encoding, and relay encoding: Same as for DF

Block	1	2	3	$\cdots$	$b-1$	b
X_1	$x_1^n(m_1\|1)$	$x_1^n(m_2\|m_1)$	$x_1^n(m_3\|m_2)$	$\cdots$	$x_1^n(m_{b-1}\|m_{b-2})$	$x_1^n(1\|m_{b-1})$
Y_2	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\cdots$	$\tilde{m}_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(\tilde{m}_1)$	$x_2^n(\tilde{m}_2)$	$\cdots$	$x_2^n(\tilde{m}_{b-2})$	$x_2^n(\tilde{m}_{b-1})$
Y_3	$\emptyset$	$\hat{m}_1$	$\hat{m}_2$	$\cdots$	$\hat{m}_{b-2}$	$\hat{m}_{b-1}$

- Successful relay decoding: $R < I(X_1; Y_2 | X_2) - \delta(\epsilon)$
- Decoding:
  - At the end of block $j+1$, find the unique $\hat{m}_j$ such that:
    $(x_1^n(\hat{m}_j | \hat{m}_{j-1}), x_2^n(\hat{m}_{j-1}), y_3^n(j)) \in \mathcal{T}_\epsilon^{(n)}$ and $(x_2^n(\hat{m}_j), y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$ simultaneously
  - Assuming $\hat{m}_{j-1}$ is correct, successful if $R < I(X_1, X_2; Y_3) - \delta(\epsilon)$

Noisy network coding

- Compress–forward for DM-RC can be extended to DM-MN

Theorem 18.3

$$C \geq \max \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, k \in \mathcal{S}^c} \left(I(X(\mathcal{S}); \hat{Y}(\mathcal{S}^c), Y_k | X(\mathcal{S}^c)) - I(Y(\mathcal{S}); \hat{Y}(\mathcal{S}) | X^N, \hat{Y}(\mathcal{S}^c), Y_k) \right),$$

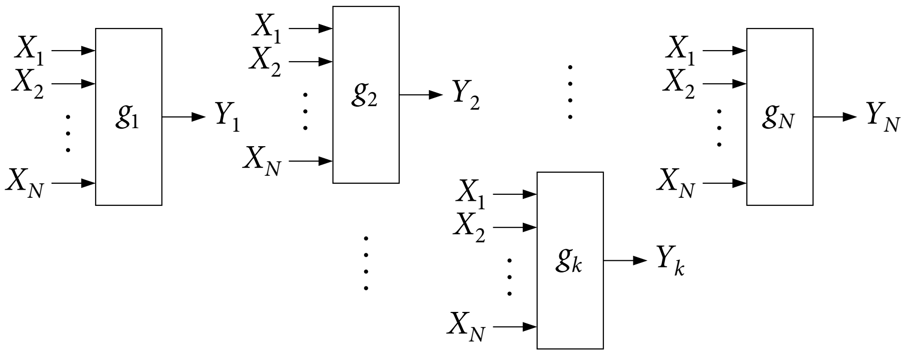
where the maximum is over all $\prod_{k=1}^{N} p(x_k) p(\hat{y}_k | y_k, x_k)$

- Comparison to the cutset bound

$$C \leq \max_{p(x^N)} \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, k \in \mathcal{S}^c} I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

  - Compressed version $\hat{Y}(\mathcal{S}^c)$ instead of $Y(\mathcal{S}^c)$
  - Rate penalty for conveying the compressed version
  - Maximum over product pmfs

Deterministic multicast network



- Cutset bound reduces to:

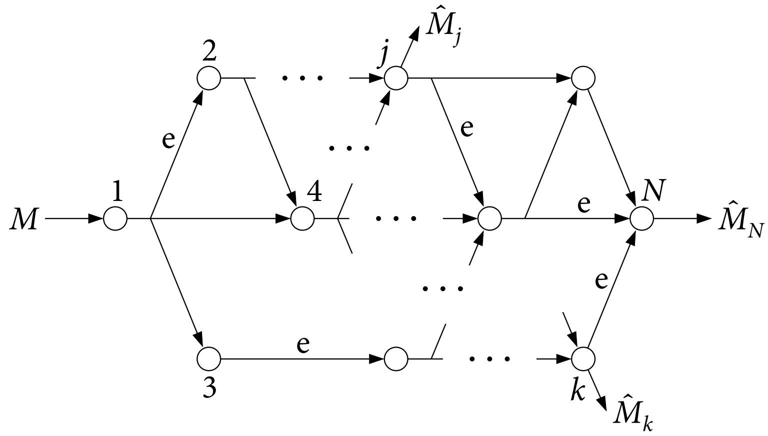
$$C \leq \max_{p(x^N)} \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, k \in \mathcal{S}^c} H(Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

- Lower bound (Avestimehr–Diggavi–Tse 2011): Use NNC with $\hat{Y}_k = Y_k$

$$C \geq \max_{\prod_{k=1}^{N} p(x_k)} \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, k \in \mathcal{S}^c} H(Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

- This bound (hence NNC) is tight for:
  - Graphical MN (Ahlswede–Cai–Li–Yeung 2000) with feedback and broadcasting
  - Deterministic MN with no interference (Ratnakar–Kramer 2006)
  - Finite-field deterministic MN (Avestimehr–Diggavi–Tse 2011)

Wireless erasure multicast network



- Hypergraph with node k broadcasting X_k and receiving $Y_k = (Y_{kj})$, where

$$Y_{kj} = \begin{cases} \mathrm{e} & \text{with probability } p_{kj}, \\ X_j & \text{with probability } 1 - p_{kj} \end{cases}$$

- Capacity (Dana–Gowaikar–Palanki–Hassibi–Effros 2006): Cutset = NNC

$$C = \min_{j \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, j \in \mathcal{S}^c} \sum_{k \in \mathcal{S}: \mathcal{N}_k \cap \mathcal{S}^c \neq \emptyset} \Bigl(1 - \prod_{l \in \mathcal{N}_k \cap \mathcal{S}^c} p_{lk}\Bigr) C_k$$

Achievability of NNC lower bound (for DM-RC)

- We use several new ideas beyond compress–forward for DM-RC
  - Sender: Transmission of same message $m \in [1:2^{nbR}]$ over b blocks
  - Relays: Compression of Y_j^n by $\hat{Y}_j^n$ without binning
  - Receivers: Simultaneous nonunique decoding of message and all compression indices

- Codebook generation:
  - Fix $p(x_1)p(x_2)p(\hat{y}_2|y_2, x_2)$ that attains the lower bound
  - For each $j \in [1:b]$, generate 2^{nbR} sequences $x_1^n(j, m) \sim \prod_{i=1}^n p_{X_1}(x_{1i})$, $m \in [1:2^{nbR}]$
  - Generate 2^{nR_2} sequences $x_2^n(l_{j-1}) \sim \prod_{i=1}^n p_{X_2}(x_{2i})$, $l_{j-1} \in [1:2^{nR_2}]$
  - For each $l_{j-1} \in [1:2^{nR_2}]$, conditionally independently generate 2^{nR_2} sequences $\hat{y}_2^n(l_j|l_{j-1}) \sim \prod_{i=1}^n p_{\hat{Y}_2|X_2}(\hat{y}_{2i}|x_{2i}(l_{j-1}))$, $l_j \in [1:2^{nR_2}]$

- $\mathcal{C}_j = \{(x_1^n(j, m), x_2^n(l_{j-1}), \hat{y}_2^n(l_j|l_{j-1})) : m \in [1:2^{nbR}], l_{j-1}, l_j \in [1:2^{nR_2}]\}$, $j \in [1:b]$

- Encoding:
  - To send $m \in [1:2^{nbR}]$, transmit $x_1^n(j, m)$ in block j

Achievability of NNC lower bound (for DM-RC)

Block	1	2	3	$\cdots$	$b-1$	b
X_1	$x_1^n(1,m)$	$x_1^n(2,m)$	$x_1^n(3,m)$	$\cdots$	$x_1^n(b-1,m)$	$x_1^n(b,m)$
Y_2	$\hat{y}_2^n(l_1\|1), l_1$	$\hat{y}_2^n(l_2\|l_1), l_2$	$\hat{y}_2^n(l_3\|l_2), l_3$	$\cdots$	$\hat{y}_2^n(l_{b-1}\|l_{b-2}), l_{b-1}$	$\hat{y}_2^n(l_b\|l_{b-1}), l_b$
X_2	$x_2^n(1)$	$x_2^n(l_1)$	$x_2^n(l_2)$	$\cdots$	$x_2^n(l_{b-2})$	$x_2^n(l_{b-1})$
Y_3	$\emptyset$	$\emptyset$	$\emptyset$	$\cdots$	$\emptyset$	$\hat{m}$

- Relay encoding:
  - At the end of block j, find l_j such that $(y_2^n(j), \hat{y}_2^n(l_j|l_{j-1}), x_2^n(l_{j-1})) \in \mathcal{T}_{\epsilon'}^{(n)}$ (by covering lemma successful if $R_2 > I(Y_2; \hat{Y}_2|X_2) + \delta(\epsilon'))$
  - In block $j+1$, transmit $x_2^n(l_j)$
- Decoding:
  - At the end of block b, find the unique $\hat{m}$ such that $(x_1^n(j,\hat{m}), x_2^n(l_{j-1}), \hat{y}_2^n(l_j|l_{j-1}), y_3^n(j)) \in \mathcal{T}_\epsilon^{(n)}$ for all $j \in [1:b]$ for some $l_1, l_2, \ldots, l_b$ (successful if $R < \min\{I(X_1; \hat{Y}_2, Y_3|X_2),\ I(X_1, X_2; Y_3) + I(\hat{Y}_2; X_1, Y_3|X_2) - R_2\}$)

Extension to multiple messages

- Node $j \in [1:N]$ sends M_j to $\mathcal{D}_j \subseteq [1:N]$
- Cutset bound can be extended

Theorem 18.4

If $(R_1, \ldots, R_N)$ is achievable, then

$$\sum_{j\in\mathcal{S}:\mathcal{D}_j\cap\mathcal{S}^c\neq\emptyset} R_j \leq I(X(\mathcal{S}); Y(\mathcal{S}^c)|X(\mathcal{S}^c))$$

for all $\mathcal{S} \subset [1:N]$ such that $\mathcal{S}^c \cap \bigcup_{j\in\mathcal{S}} \mathcal{D}_j \neq \emptyset$ for some pmf $p(x^N)$

- NNC can be extended via interference channel schemes (NIT 18.4)

Gaussian multimessage network

- N-node Gaussian network: $Y^N = GX^N + Z^N$
  - Z^N: i.i.d. $\mathrm{N}(0,1)$
  - $G \in \mathbb{R}^{N\times N}$: the channel gain matrix (gain G_{jk} from k to j)
  - Average power constraint P on each sender

Theorem 19.1 (Cutset bound)

If $(R_1,\ldots,R_N)$ is achievable, then

$$\sum_{j\in\mathcal{S}:\mathcal{D}_j\cap\mathcal{S}^c\neq\emptyset} R_j \le \frac{1}{2}\log|I + G(\mathcal{S})K(\mathcal{S}|\mathcal{S}^c)G^T(\mathcal{S})|$$

for all $\mathcal{S}$ such that $\mathcal{S}^c \cap \bigcup_{j\in\mathcal{S}} \mathcal{D}_j \neq \emptyset$ for some $K \succeq 0$ with $K_{jj} \le P$, $j\in[1:N]$

Notation:

$$\begin{bmatrix} Y(\mathcal{S}) \\ Y(\mathcal{S}^c)\end{bmatrix} = \begin{bmatrix} G'(\mathcal{S}) & G(\mathcal{S}^c) \\ G(\mathcal{S}) & G'(\mathcal{S}^c)\end{bmatrix}\begin{bmatrix} X(\mathcal{S}) \\ X(\mathcal{S}^c)\end{bmatrix} + \begin{bmatrix} Z(\mathcal{S}) \\ Z(\mathcal{S}^c)\end{bmatrix}$$

$K(\mathcal{S}|\mathcal{S}^c)$: conditional covariance matrix of $X(\mathcal{S})$ given $X(\mathcal{S}^c)$

Multimessage multicast

- Suppose $\mathcal{D}_j = \mathcal{D}$, $j\in[1:N]$
- Cutset bound: If $(R_1,\ldots,R_N)$ is achievable, then

$$\sum_{j\in\mathcal{S}} R_j \le \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})G^T(\mathcal{S})\right| + \frac{N}{2}\log 3$$

- Noisy network coding: $(R_1,\ldots,R_N)$ is achievable if

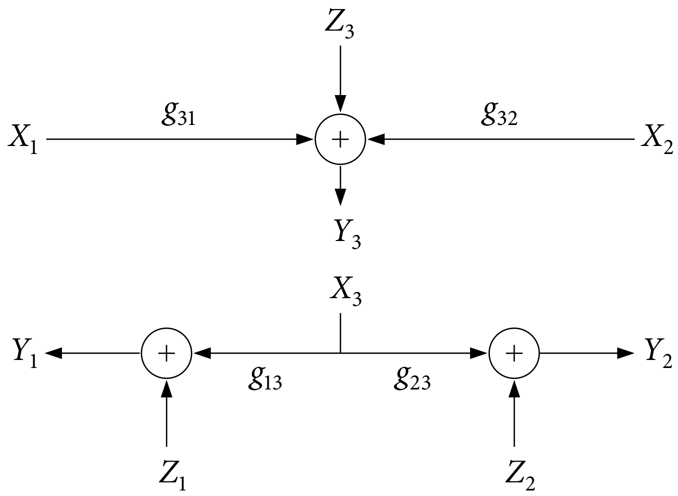
$$\sum_{j\in\mathcal{S}} R_j < \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})G^T(\mathcal{S})\right| - \frac{N}{2}$$

(set $\hat{Y}_k = Y_k + \hat{Z}_k$, where $\hat{Z}_k \sim \mathrm{N}(0,1)$)

Theorem 19.2 (Constant gap)

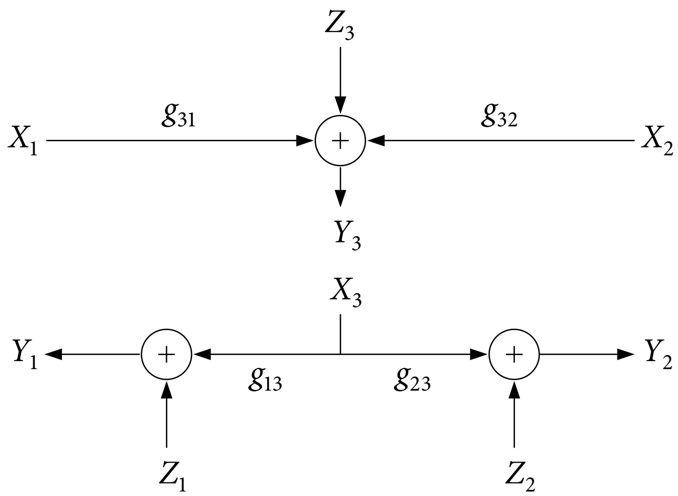
If $(R_1,\ldots,R_N)$ is in the cutset bound, then $(R_1-\Delta,\ldots,R_N-\Delta)$, where $\Delta = (N/2)\log 6$, is achievable via NNC

Example: Gaussian two-way relay channel



- $Z_1, Z_2, Z_3 \sim \mathrm{N}(0,1)$ independent of each other
- Power constraints P on each sender
- SNRs: $S_{jk} = g_{jk}^2 P$
- Nodes 1 and 2 wish to exchange M_1 and M_2

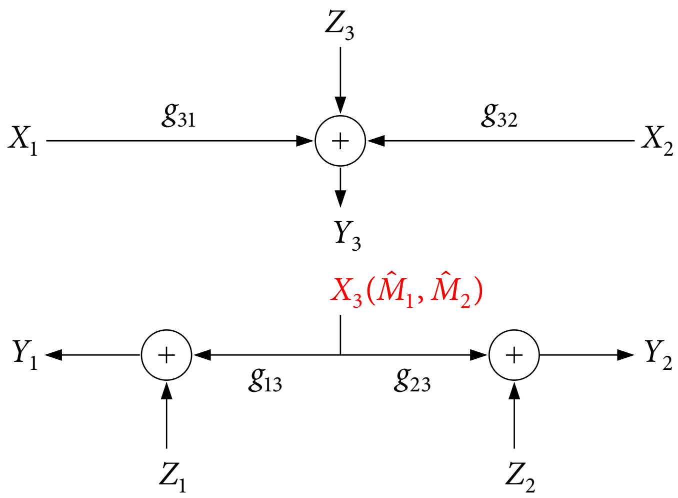
Example: Gaussian two-way relay channel



- Capacity region is not known in general
- Cutset bound: If (R_1, R_2) is achievable, then

$$R_1 \le \min\{\mathsf{C}(S_{31}),\ \mathsf{C}(S_{23})\},$$
$$R_2 \le \min\{\mathsf{C}(S_{32}),\ \mathsf{C}(S_{13})\}$$

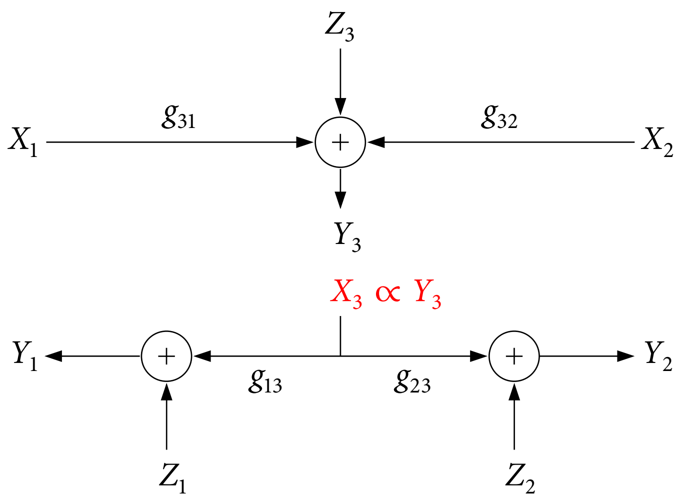
Example: Gaussian two-way relay channel



- Decode–forward inner bound: (R_1, R_2) is achievable if

$$\begin{aligned} R_1 &< \min\{\mathsf{C}(S_{31}),\ \mathsf{C}(S_{23})\},\\ R_2 &< \min\{\mathsf{C}(S_{32}),\ \mathsf{C}(S_{13})\},\\ R_1 + R_2 &< \mathsf{C}(S_{31} + S_{32}) \end{aligned}$$

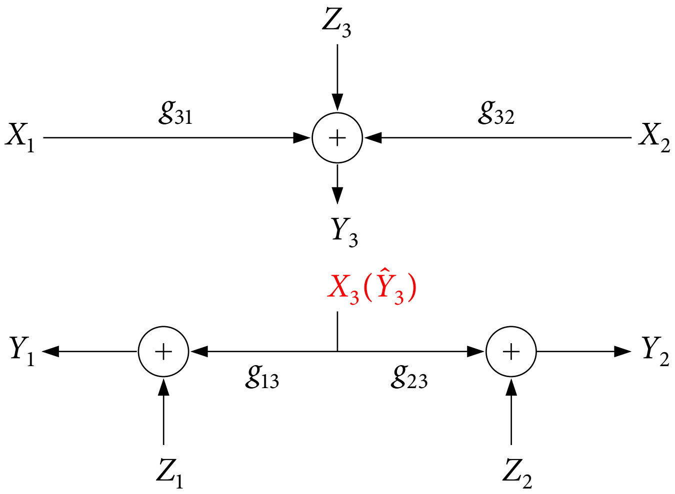
Example: Gaussian two-way relay channel



- Amplify–forward inner bound: $X_3 \propto Y_3$; (R_1, R_2) is achievable if

$$\begin{aligned} R_1 &< \mathsf{C}\left(\frac{S_{23}S_{31}}{1 + S_{23} + S_{31} + S_{32}}\right),\\ R_2 &< \mathsf{C}\left(\frac{S_{13}S_{32}}{1 + S_{13} + S_{31} + S_{32}}\right) \end{aligned}$$

Example: Gaussian two-way relay channel

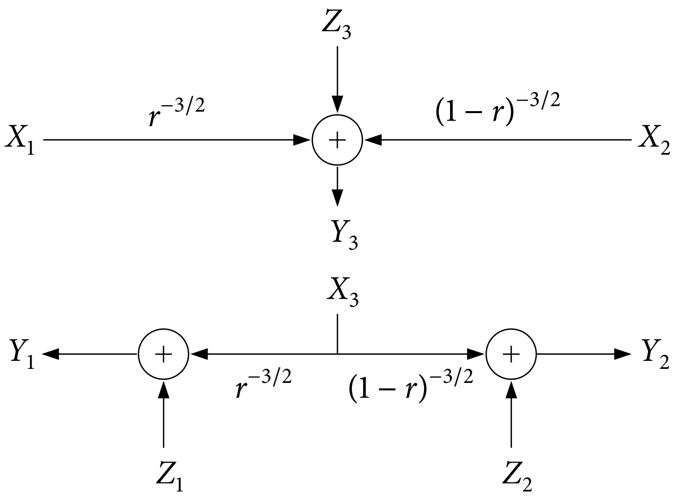


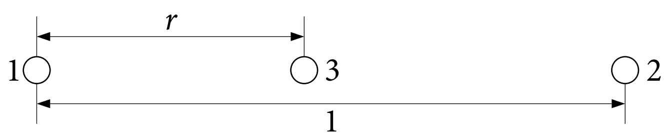
- Noisy network coding inner bound: (R_1, R_2) is achievable if

$$R_1 < \min\{\mathsf{C}(S_{31}/(1+\sigma^2)), \mathsf{C}(S_{23}) - \mathsf{C}(1/\sigma^2)\},$$
$$R_2 < \min\{\mathsf{C}(S_{32}/(1+\sigma^2)), \mathsf{C}(S_{13}) - \mathsf{C}(1/\sigma^2)\}$$

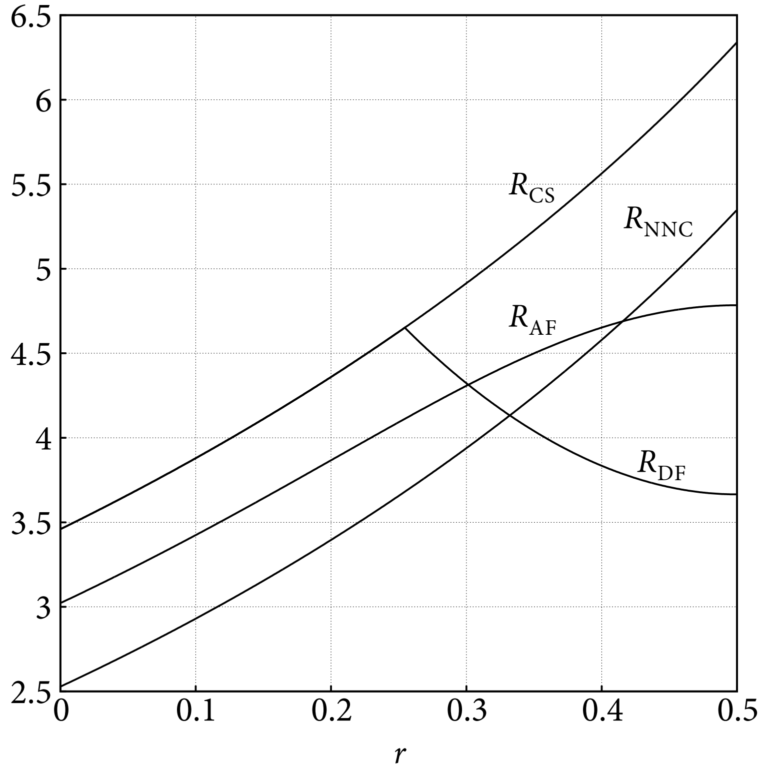
  - Achieves the capacity region within $1/2$-bit per dimension
  - DF and AF have unbounded gap

Example





Comparison of the bounds on the sum-capacity



Summary

- General network models that include graphical networks, RC, and single-hop networks with feedback as special cases
- Cutset bounds for the DMN and Gaussian networks
- Sliding window decoding for network decode–forward
- Noisy network coding:
  - Same message is sent multiple times using independent codebooks
  - No binning
  - Simultaneous nonunique decoding without requiring the recovery of compression bin indices
  - Includes CF for the relay channel and network coding for graphical networks
  - Achieves within constant gap of cutset bound for Gaussian multimessage multicast

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