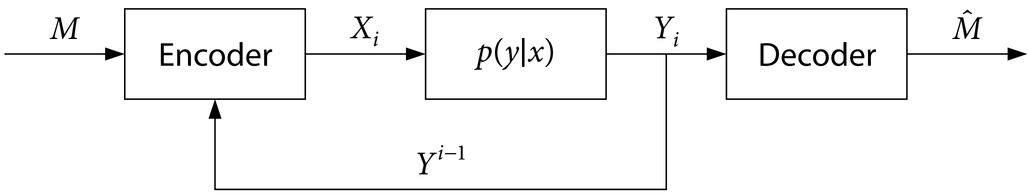
Lecture #11 Interactive Channel Coding

(Reading: NIT 17.1–17.4)

- DMC with feedback
- MAC with feedback
- BC with feedback
- RC with feedback

DMC with feedback



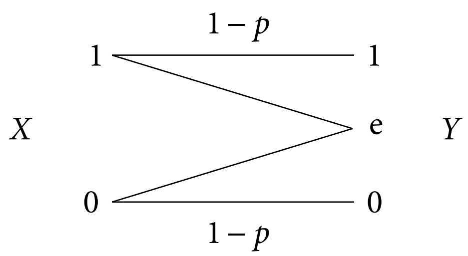
- DMC $p(y|x)$ with capacity C
- Feedback ($x_i(M, Y^{i-1})$) does not increase the capacity (Shannon 1956), i.e.,

$$C_{\text{FB}} = \max_{p(x)} I(X; Y) = C$$

- Feedback can help communication in several important ways
  - Simplify coding and improve reliability (Schalkwijk–Kailath 1966)
  - Increase the capacity of channels with memory (Butman 1969)
  - Enlarge the capacity region of multiuser channels (Gaarder–Wolf 1975)
- Insights into interactive multi-way communication

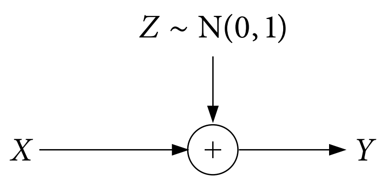
Iterative refinement

- Basic idea:
  - First send the message uncoded
  - Then iteratively refine the receiver's knowledge about it
- Simple example (Binary erasure channel with feedback):



- More sophisticated examples:
  - Schalkwijk–Kailath coding scheme for the Gaussian channel (1966)
  - Horstein's coding scheme for BSC (1963)
  - Posterior matching scheme (Shayevitz–Feder 2011)
  - Block feedback coding scheme (Weldon 1963, Ahlswede 1973, Ooi–Wornell 1998)

Gaussian channel with feedback



- Expected average transmitted power constraint

$$\sum_{i=1}^{n} \mathsf{E}(x_i^2(m, Y^{i-1})) \le nP, \quad m \in [1:2^{nR}]$$

- Schalkwijk–Kailath coding scheme (1966):
  - Divide the interval $[-\sqrt{P}, \sqrt{P}]$ into 2^{nR} equal-length message intervals
  - Represent each message $m \in [1:2^{nR}]$ by the midpoint of its interval
  - At time 0, encoder transmits $X_0 = \theta(m)$ and learns $Z_0 = Y_0 - X_0$
  - At time 1, it transmits $X_1 = \sqrt{P}\, Z_0$
  - For $i \in [2:n]$, it transmits $X_i \propto Z_0 - \mathsf{E}(Z_0 | Y^{i-1})$ (scaled to meet power constraint)
  - Upon receiving Y^n, the decoder finds the closest message point $\theta(\hat{m})$ to

$$\hat{\Theta}_n = Y_0 - \mathsf{E}(Z_0 | Y^n) = \theta(m) + Z_0 - \mathsf{E}(Z_0 | Y^n)$$

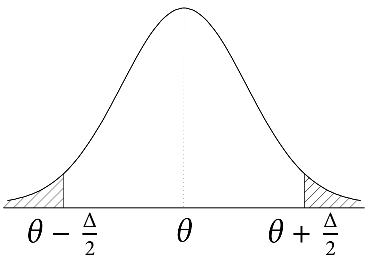
Gaussian channel with feedback

- By linearity and the orthogonality principle, $Y_1, Y_2, \ldots, Y_n$ are i.i.d. $\mathrm{N}(0, P+1)$
- By expanding $I(Z_0; Y^n)$ in two ways, it can be shown that

$$\mathrm{Var}(Z_0 \mid Y^n) = 2^{-2n\,\mathsf{C}(P)}$$

- Thus, $\hat{\Theta}_n = \theta(m) + Z_0 - \mathsf{E}(Z_0|Y^n) \sim \mathrm{N}(\theta(m), 2^{-2nC(P)})$
- Since the distance to the nearest message point is $\Delta = 2\sqrt{P} \cdot 2^{-nR}$,

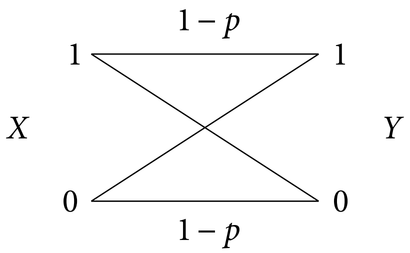
$$P_e^{(n)} \le 2Q\left(\frac{\sqrt{P} \cdot 2^{-nR}}{2^{-nC(P)}}\right) = Q(2^{n(\mathsf{C}(P)-R)}\sqrt{P})$$

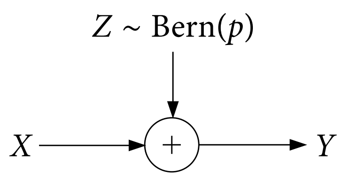


- If $R < C$, then $P_e^{(n)}$ decays double-exponentially fast in n !

Block feedback coding scheme

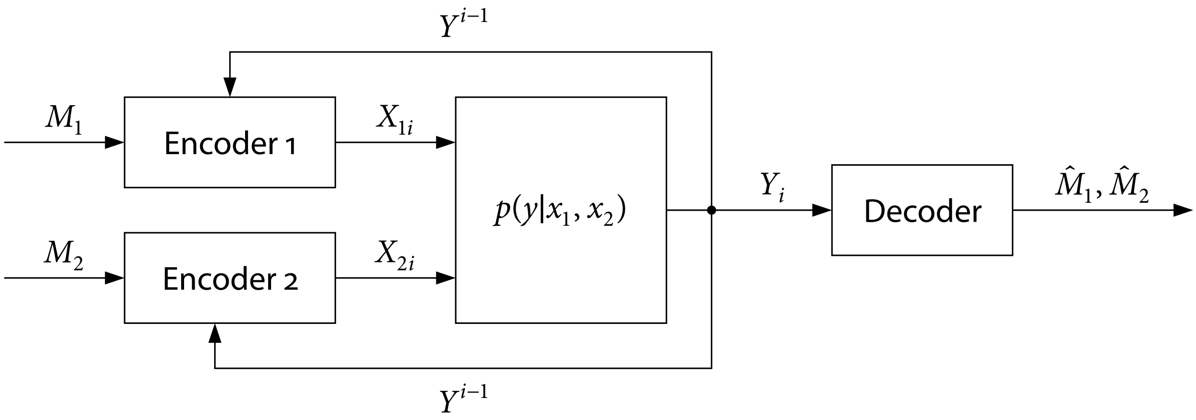
- Implementation of iterative refinement at the block level (Weldon 1963)
- Example (BSC with feedback):





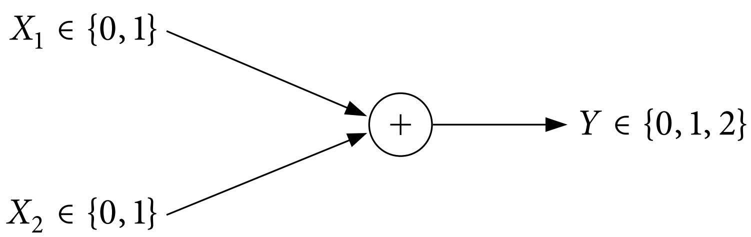
  - Transmit k bits uncoded and learn the error vector via feedback
  - Compress it using $kH(p)$ bits and transmit the compression index uncoded
  - Communicate the error about the error ($kH^2(p)$ bits)
  - Communicate the error about the error about the error ($kH^3(p)$ bits)
  - Communicate the error about the error about the error about …
- Achievable rate: $k/(k + kH(p) + kH^2(p) + kH^3(p) + \cdots) = 1 - H(p)$

MAC with feedback



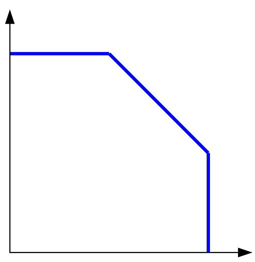
- Cooperative transmission: $x_{1i}(M_1, Y^{i-1}), x_{2i}(M_2, Y^{i-1})$
- Capacity region with feedback $\mathscr{C}_{\text{FB}}$ is not known in general
- Feedback can enlarge the capacity region

Binary erasure MAC



- Capacity region without feedback:

$$R_1 \le 1,$$
$$R_2 \le 1,$$
$$R_1 + R_2 \le 3/2$$



- Block feedback coding scheme (Gaarder–Wolf 1975):
  - k uncoded transmissions + $k/4$ + $k/4$ one-sided retransmissions $\Rightarrow R_{\text{sym}} = 2/3$
  - k uncoded transmissions + $k/4$ two-sided retransmissions + $k/16 + \cdots \Rightarrow R_{\text{sym}} = 3/4$
  - k uncoded transmissions + $k/(2 \log 3)$ cooperative retransmissions $\Rightarrow R_{\text{sym}} = 0.7602$
- $R^*_{\text{sym}} = 0.7911$ (Cover–Leung 1981, Willems 1982)

Cover–Leung inner bound

Theorem 17.1 (Cover–Leung 1981)

(R_1, R_2) is achievable for the DM-MAC $p(y|x_1, x_2)$ with feedback if

$$\begin{aligned} R_1 &< I(X_1; Y | X_2, U), \\ R_2 &< I(X_2; Y | X_1, U), \\ R_1 + R_2 &< I(X_1, X_2; Y) \end{aligned}$$

for some pmf $p(u)p(x_1|u)p(x_2|u)$

- Capacity region without feedback: The set of (R_1, R_2) such that

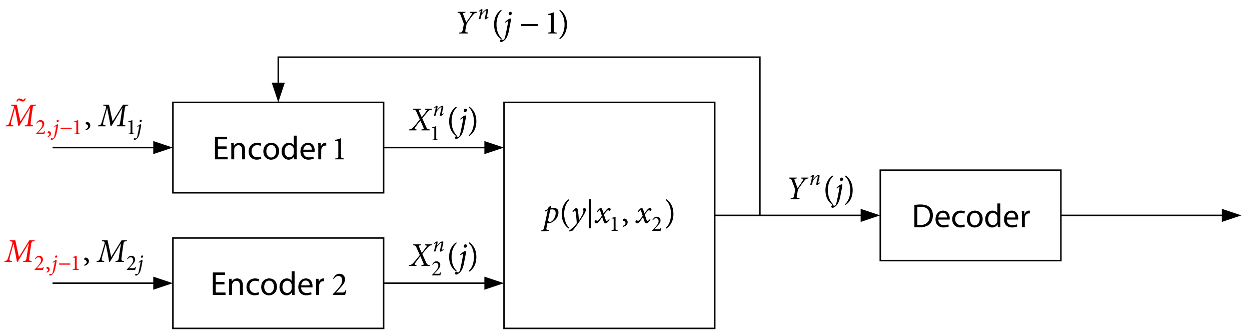
$$\begin{aligned} R_1 &\le I(X_1; Y | X_2, Q), \\ R_2 &\le I(X_2; Y | X_1, Q), \\ R_1 + R_2 &\le I(X_1, X_2; Y | Q) \end{aligned}$$

  for some pmf $p(q)p(x_1|q)p(x_2|q)$

- Cover–Leung inner bound is tight when X_1 is a function of (X_2, Y) (Willems 1982)

Proof of achievability (Zeng–Kuhlmann–Buzo 1989)

- Use block Markov coding scheme and backward decoding



Block	1	2	$\cdots$	$b-1$	b
X_2	$x_2^n(m_{21}\|1)$	$x_2^n(m_{22}\|m_{21})$	$\cdots$	$x_2^n(m_{2,b-1}\|m_{2,b-2})$	$x_2^n(1\|m_{2,b-1})$
X_1	$x_1^n(m_{11}\|1)$	$x_1^n(m_{12}\|\tilde{m}_{21})$	$\cdots$	$x_1^n(m_{1,b-1}\|\tilde{m}_{2,b-2})$	$x_1^n(m_{1b}\|\tilde{m}_{2,b-1})$
(X_1, Y)	$\tilde{m}_{21} \to$	$\tilde{m}_{22} \to$	$\cdots$	$\tilde{m}_{2,b-1}$	$\emptyset$
Y	$\hat{m}_{11}$	$\leftarrow \hat{m}_{12}, \hat{m}_{21}$	$\cdots$	$\leftarrow \hat{m}_{1,b-1}, \hat{m}_{2,b-2}$	$\leftarrow \hat{m}_{1b}, \hat{m}_{2,b-1}$

Cooperative outer bound

- If (R_1, R_2) is achievable, it must satisfy the inequalities:

$$\begin{aligned} R_1 &\le I(X_1; Y|X_2), \\ R_2 &\le I(X_2; Y|X_1), \\ R_1 + R_2 &\le I(X_1, X_2; Y) \end{aligned}$$

  for some pmf $p(x_1, x_2)$

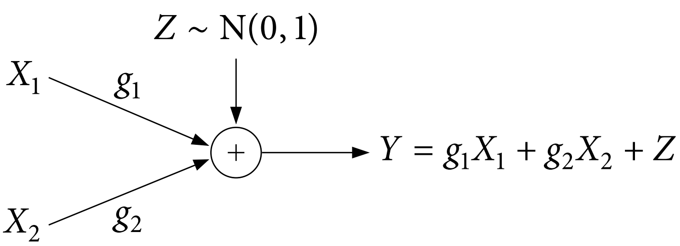
- Cover–Leung inner bound

$$\begin{aligned} R_1 &< I(X_1; Y|X_2, U), \\ R_2 &< I(X_2; Y|X_1, U), \\ R_1 + R_2 &< I(X_1, X_2; Y) \end{aligned}$$

  for some pmf $p(u)p(x_1|u)p(x_2|u)$

- Are they equal?

Gaussian MAC



- Cover–Leung inner bound: (R_1, R_2) is achievable if

$$R_1 \le \mathsf{C}(\bar{\alpha}_1 S_1), \quad R_2 \le \mathsf{C}(\bar{\alpha}_2 S_2), \quad R_1 + R_2 \le \mathsf{C}\left(S_1 + S_2 + 2\sqrt{\alpha_1 \alpha_2 S_1 S_2}\right)$$

  for some $\alpha_1, \alpha_2 \in [0, 1]$

- $\mathscr{C}_{\text{FB}}$ is larger than Cover–Leung and coincides with outer bound!

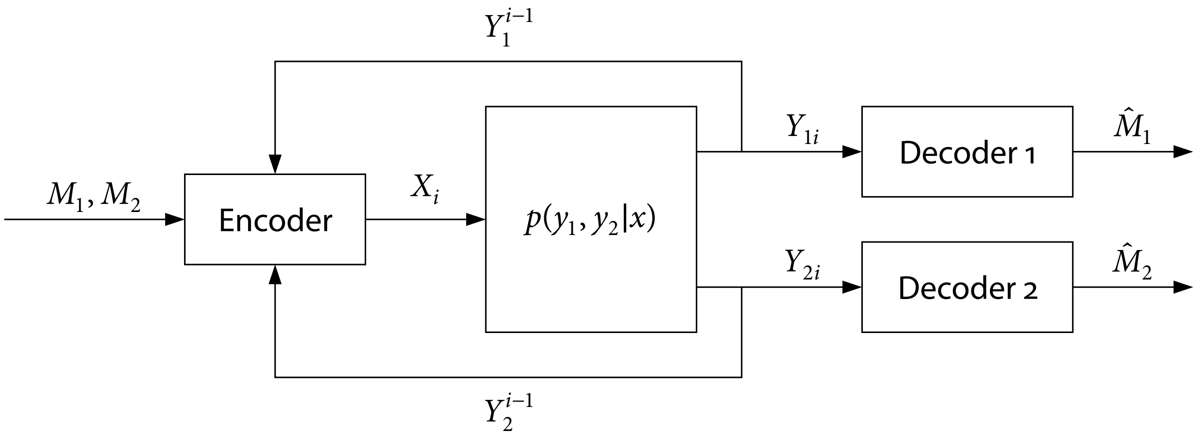
Theorem 17.2 (Ozarow 1984)

$\mathscr{C}_{\text{FB}}$ is the set of (R_1, R_2) such that

$$R_1 \le \mathsf{C}((1-\rho^2)S_1), \quad R_2 \le \mathsf{C}((1-\rho^2)S_2), \quad R_1 + R_2 \le \mathsf{C}(S_1 + S_2 + 2\rho\sqrt{S_1 S_2})$$

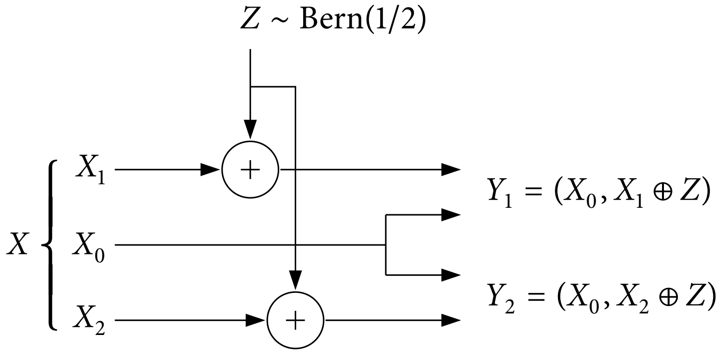
for some $\rho \in [0, 1]$

BC with feedback



- Receivers operate separately (regardless of feedback)
- Physically degraded BC $p(y_1|x)p(y_2|y_1)$:
  - Feedback does not enlarge the capacity region (El Gamal 1978)
- How can feedback help?

Dueck's example (1980)

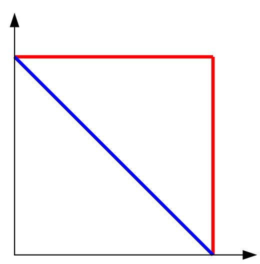


- Capacity region without feedback:

$$\{(R_1, R_2): R_1 + R_2 \leq 1\}$$

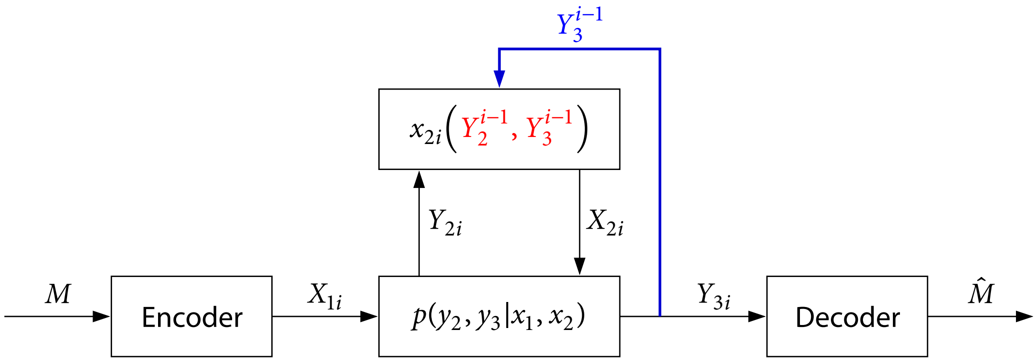
- Capacity region with feedback:

$$\{(R_1, R_2): R_1 \leq 1,\ R_2 \leq 1\}$$



- Feedback helps by letting the encoder broadcast common channel information

RC with feedback



- Feedback turns RC into physically degraded RC $p(y_2', y_3|x_1, x_2)$, $Y_2' = (Y_2, Y_3)$

Theorem 17.3 (Cover–El Gamal 1979)

$$C_{\mathrm{FB}} = \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3 \mid X_2)\}$$

- Achieved using decode–forward (cutest bound holds with feedback)
- A rare example where capacity is not known, but feedback capacity is known

Summary

- Feedback can simplify coding and improve reliability
- Iterative refinement:
  - Schalkwijk–Kailath scheme for the Gaussian channel
  - Block feedback scheme for the DMC
- Feedback can enlarge the capacity region of multiuser channels:
  - Can induce cooperation between the senders
  - Cover–Leung inner bound
  - Can provide channel information to be broadcast to multiple receivers
  - The cutset bound is achievable for the relay channel with feedback

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