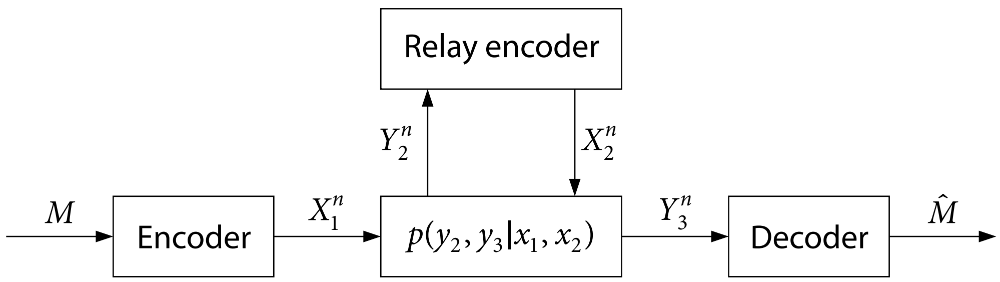
Lecture #10 Relay Channels

(Reading: NIT 16.1–16.7)

- Discrete memoryless relay channel
- Cutset upper bound
- Decode–forward
- Compress–forward
- Gaussian relay channel

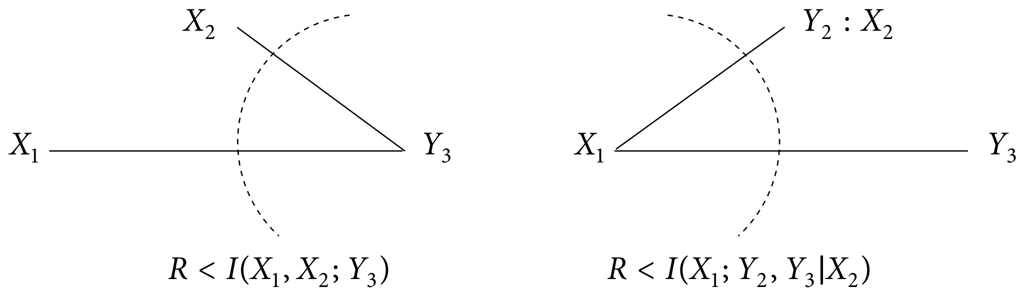
Point-to-point communication system with a relay



- DM relay channel (RC) $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_2, y_3|x_1, x_2), \mathcal{Y}_2 \times \mathcal{Y}_3)$
- A $(2^{nR}, n)$ code:
  - Message set: $[1 : 2^{nR}]$
  - Encoder: $x_1^n(m)$
  - Relay encoder: $x_{2i}(y_2^{i-1})$, $i \in [1 : n]$
  - Decoder: $\hat{m}(y_3^n)$
- $P_e^{(n)}$, achievability, C: Defined as for the DMC
- Capacity is not known in general

Cutset upper bound

- Given sequence of codes with $P_e^{(n)} \to 0$, $\exists\, p(x_1, x_2)$ such that:



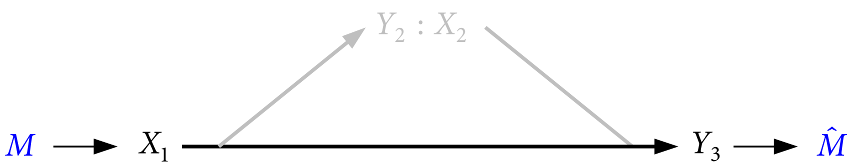
Cooperative MAC bound   Cooperative BC bound

Cutset upper bound (Cover–El Gamal 1979)

$$C \le \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3 | X_2)\}$$

- Tight for almost all classes of relay channels with known capacity
- But, is not tight in general (Example 16.2)

Direct transmission



- Relay takes no active role in communication

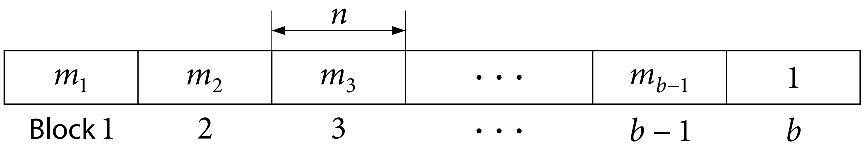
Direct-transmission lower bound

$$C \ge \max_{p(x_1),\, x_2} I(X_1; Y_3 | X_2 = x_2)$$

- Tight for reversely degraded DM-RC: $X_1 \to (Y_3, X_2) \to Y_2$

Multihop

- All communication goes through the relay
- Block Markov coding: Send $b-1$ messages over b n-transmission blocks



Multihop

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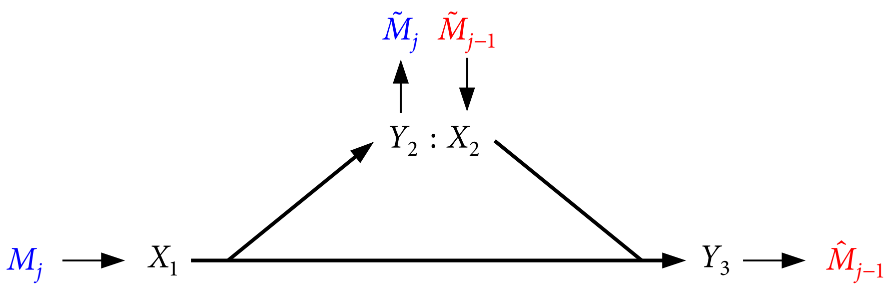


Multihop lower bound

$$C \geq \max_{p(x_1)p(x_2)} \min\{I(X_2; Y_3),\ I(X_1; Y_2|X_2)\}$$

- Tight for a cascade of two DMCs: $p(y_2, y_3|x_1, x_2) = p(y_2|x_1)p(y_3|x_2)$

$$C = \min\left\{\max_{p(x_2)} I(X_2; Y_3),\ \max_{p(x_1)} I(X_1; Y_2)\right\}$$

Proof of achievability

- Codebook generation: Fix $p(x_1)p(x_2)$ that attains bound. For each $j \in [1:b]$
  - Independently generate 2^{nR} sequences $x_1^n(m_j) \sim \prod_{i=1}^n p_{X_1}(x_{1i})$, $m_j \in [1:2^{nR}]$
  - Independently generate 2^{nR} sequences $x_2^n(m_{j-1}) \sim \prod_{i=1}^n p_{X_2}(x_{2i})$, $m_{j-1} \in [1:2^{nR}]$
- Codebooks: $\mathcal{C}_j = \{(x_1^n(m_j), x_2^n(m_{j-1})) : m_{j-1}, m_j \in [1:2^{nR}]\}$, $j \in [1:b]$
- Encoding:
  - To send m_j in block j, transmit $x_1^n(m_j)$ from $\mathcal{C}_j$
- Relay encoding:
  - At the end of block j, find unique $\tilde{m}_j$ such that $(x_1^n(\tilde{m}_j), x_2^n(\tilde{m}_{j-1}), y_2^n(j)) \in \mathcal{T}_\epsilon^{(n)}$
  - In block $j+1$, transmit $x_2^n(\tilde{m}_j)$ from $\mathcal{C}_{j+1}$
- Decoding:
  - At the end of block $j+1$, find unique $\hat{m}_j$ such that $(x_2^n(\hat{m}_j), y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$

Analysis of the probability of error

- Consider $\mathsf{P}(\mathcal{E}(j))$ for $M_j = 1$
- Let $\tilde{M}_j$ be the relay's decoded message at the end of block j
- Note that

$$\begin{aligned}\mathsf{P}(\mathcal{E}(j)) &= \mathsf{P}\{\hat{M}_j \neq 1\} \\ &\leq \mathsf{P}\{\tilde{M}_j \neq 1\} + \mathsf{P}\{\hat{M}_j \neq \tilde{M}_j\}\end{aligned}$$

- Error events:

$$\begin{aligned}\tilde{\mathcal{E}}_1(j) &= \{(X_1^n(1), X_2^n(\tilde{M}_{j-1}), Y_2^n(j)) \notin \mathcal{T}_\epsilon^{(n)}\} \\ \tilde{\mathcal{E}}_2(j) &= \{(X_1^n(m_j), X_2^n(\tilde{M}_{j-1}), Y_2^n(j)) \in \mathcal{T}_\epsilon^{(n)} \text{ for some } m_j \neq 1\} \\ \mathcal{E}_1(j) &= \{(X_2^n(\tilde{M}_j), Y_3^n(j+1)) \notin \mathcal{T}_\epsilon^{(n)}\} \\ \mathcal{E}_2(j) &= \{(X_2^n(m_j), Y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)} \text{ for some } m_j \neq \tilde{M}_j\}\end{aligned}$$

- Thus, by the union of the events bound

$$\mathsf{P}(\mathcal{E}(j)) \leq \mathsf{P}(\tilde{\mathcal{E}}_1(j)) + \mathsf{P}(\tilde{\mathcal{E}}_2(j)) + \mathsf{P}(\mathcal{E}_1(j)) + \mathsf{P}(\mathcal{E}_2(j))$$

Analysis of the probability of error

- Error events:

$$\tilde{\mathcal{E}}_1(j) = \{(X_1^n(1), X_2^n(\tilde{M}_{j-1}), Y_2^n(j)) \notin \mathcal{T}_\epsilon^{(n)}\}$$
$$\tilde{\mathcal{E}}_2(j) = \{(X_1^n(m_j), X_2^n(\tilde{M}_{j-1}), Y_2^n(j)) \in \mathcal{T}_\epsilon^{(n)} \text{ for some } m_j \neq 1\}$$
$$\mathcal{E}_1(j) = \{(X_2^n(\tilde{M}_j), Y_3^n(j+1)) \notin \mathcal{T}_\epsilon^{(n)}\}$$
$$\mathcal{E}_2(j) = \{(X_2^n(m_j), Y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)} \text{ for some } m_j \neq \tilde{M}_j\}$$

- By the independence of the codebooks, $\tilde{M}_{j-1}$ is independent of $\mathcal{C}_j$
  Thus by the LLN, $\mathsf{P}(\tilde{\mathcal{E}}_1(j)) \to 0$

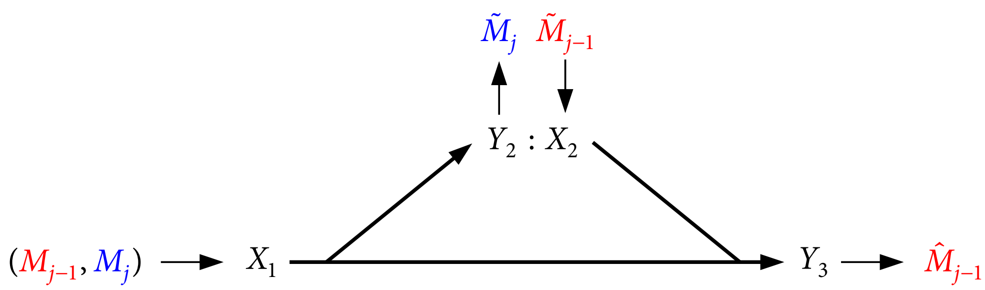
- By the same independence and the packing lemma, $\mathsf{P}(\tilde{\mathcal{E}}_2(j)) \to 0$ if $R < I(X_1; Y_2|X_2) - \delta(\epsilon)$

- By the independence of the codebooks and the LLN, $\mathsf{P}(\mathcal{E}_1(j)) \to 0$

- By the same independence and the packing lemma,
  $\mathsf{P}(\mathcal{E}_2(j)) \to 0$ if $R < I(X_2; Y_3) - \delta(\epsilon)$

Coherent multihop

- Since sender knows what relay knows, they can coherently cooperate



Coherent multihop lower bound

$$C \geq \max_{p(x_1)p(x_2)} \min\{I(X_2; Y_3),\ I(X_1; Y_2|X_2)\}$$

Proof of achievability

- Again use block Markov coding

m_1	m_2	m_3	$\cdots$	m_{b-1}	1
Block 1	2	3	$\cdots$	$b-1$	b

(Each block has length n.)

- Codebook generation: Fix $p(x_1, x_2)$ that attains bound. For $j \in [1:b]$:
  - Independently generate 2^{nR} sequences $x_2^n(m_{j-1}) \sim \prod_{i=1}^n p_{X_2}(x_{2i})$, $m_{j-1} \in [1:2^{nR}]$
  - For each $m_{j-1} \in [1:2^{nR}]$, conditionally independently generate 2^{nR} sequences $x_1^n(m_j|m_{j-1}) \sim \prod_{i=1}^n p_{X_1|X_2}(x_{1i}|x_{2i}(m_{j-1}))$, $m_j \in [1:2^{nR}]$
- Codebooks: $\mathcal{C}_j = \{(x_1^n(m_j|m_{j-1}), x_2^n(m_{j-1})) : m_{j-1}, m_j \in [1:2^{nR}]\}$, $j \in [1:b]$

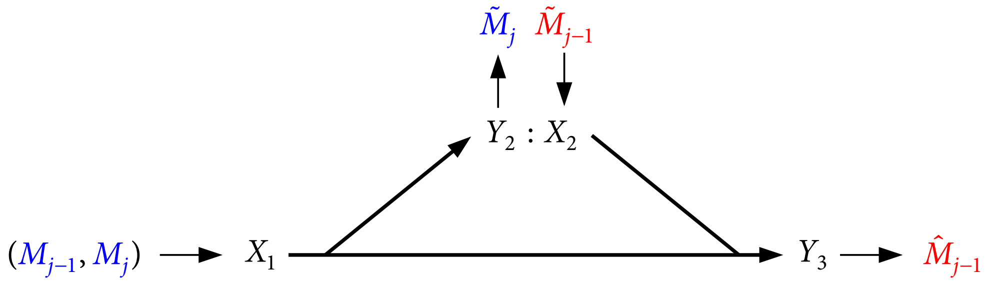
Proof of achievability

Block	1	2	3	$\ldots$	$b-1$	b
X_1	$x_1^n(m_1\|1)$	$x_1^n(m_2\|m_1)$	$x_1^n(m_3\|m_2)$	$\ldots$	$x_1^n(m_{b-1}\|m_{b-2})$	$x_1^n(1\|m_{b-1})$
Y_2	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\ldots$	$\tilde{m}_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(\tilde{m}_1)$	$x_2^n(\tilde{m}_2)$	$\ldots$	$x_2^n(\tilde{m}_{b-2})$	$x_2^n(\tilde{m}_{b-1})$
Y_3	$\emptyset$	$\hat{m}_1$	$\hat{m}_2$	$\ldots$	$\hat{m}_{b-2}$	$\hat{m}_{b-1}$

- Encoding:
  - In block j, transmit $x_1^n(m_j|m_{j-1})$ from codebook $\mathcal{C}_j$
- Relay encoding:
  - At end of block j, find unique $\tilde{m}_j$ such that $(x_1^n(\tilde{m}_j|\tilde{m}_{j-1}), x_2^n(\tilde{m}_{j-1}), y_2^n(j)) \in \mathcal{T}_\epsilon^{(n)}$ (successful if $R < I(X_1; Y_2|X_2) - \delta(\epsilon)$)
  - In block $j+1$, transmit $x_2^n(\tilde{m}_j)$ from codebook $\mathcal{C}_{j+1}$
- Decoding:
  - At end of block $j+1$, find unique message $\hat{m}_j$ such that $(x_2^n(\hat{m}_j), y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$ (successful if $R < I(X_2; Y_3) - \delta(\epsilon)$)

Decode–forward

- Also decode information from X_1 via backward decoding



Decode–forward lower bound (Cover–El Gamal 1979)

$$C \geq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2|X_2)\}$$

- Tight for a physically degraded DM-RC: $X_1 \to (Y_2, X_2) \to Y_3$

Proof of achievability (Zeng–Kuhlmann–Buzo 1989)

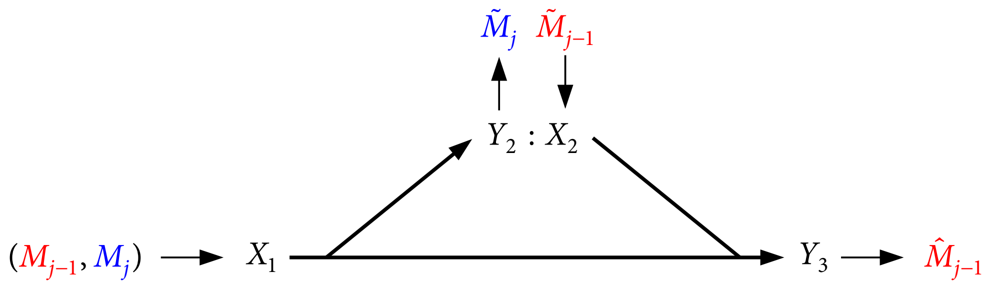
- Codebook generation, encoding, relay encoding:
  - Same as coherent multihop
  - Codebooks: $\mathcal{C}_j = \{(x_1^n(m_j|m_{j-1}), x_2^n(m_{j-1})) : m_{j-1}, m_j \in [1:2^{nR}]\}, j \in [1:b]$

Block	1	2	3	...	$b-1$	b
X_1	$x_1^n(m_1\|1)$	$x_1^n(m_2\|m_1)$	$x_1^n(m_3\|m_2)$	...	$x_1^n(m_{b-1}\|m_{b-2})$	$x_1^n(1\|m_{b-1})$
Y_2	$\tilde{m}_1 \to$	$\tilde{m}_2 \to$	$\tilde{m}_3 \to$	...	$\tilde{m}_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(\tilde{m}_1)$	$x_2^n(\tilde{m}_2)$	...	$x_2^n(\tilde{m}_{b-2})$	$x_2^n(\tilde{m}_{b-1})$
Y_3	$\emptyset$	$\hat{m}_1$	$\leftarrow \hat{m}_2$	...	$\leftarrow \hat{m}_{b-2}$	$\leftarrow \hat{m}_{b-1}$

  - Successful relay decoding: $R < I(X_1; Y_2|X_2) - \delta(\epsilon)$

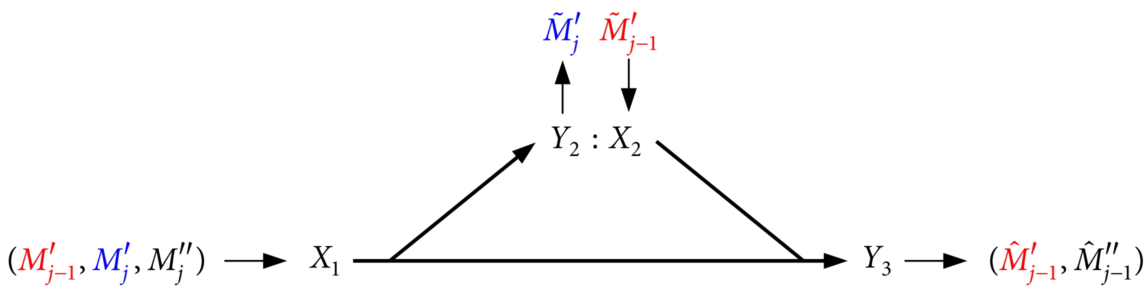
- Backward decoding (Willems–van der Meulen 1985): $\hat{m}_b = 1$
  - For $j = b-1, \ldots, 1$, find unique $\hat{m}_j$ with $(x_1^n(\hat{m}_{j+1}|\hat{m}_j), x_2^n(\hat{m}_j), y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$
  - Assuming $\hat{m}_{j+1}$ is correct, successful if $R < I(X_1, X_2; Y_3) - \delta(\epsilon)$

Decode–forward



- Decode–forward: $C \geq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2|X_2)\}$
  Cutset bound: $C \leq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(X_1; Y_2, Y_3|X_2)\}$
- Performs well when the relay is stronger than the receiver
- But worse than even direct transmission if the relay is weaker
- Solutions:
  - Partial decode–forward: Decode only part of the message
  - Compress–forward: Do not decode the message at all

Partial decode–forward



Partial decode–forward lower bound (Cover–El Gamal 1979)

$$C \geq \max_{p(u,x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(U; Y_2|X_2) + I(X_1; Y_3|X_2, U)\}$$

- Achievability outline:
  - Codebook: For $j \in [1:b]$,

$$\mathcal{C}_j = \left\{(u^n(m'_j|m'_{j-1}), x_1^n(m'_j, m''_j|m'_{j-1}), x_2^n(m'_{j-1})) : m'_{j-1}, m'_j \in [1:2^{nR'}], m''_j \in [1:2^{nR''}]\right\}$$

  - Coherent cooperation on M'_j: $R' < \min\{I(U; Y_2|X_2) - \delta(\epsilon),\ I(U, X_2; Y_3) - \delta(\epsilon)\}$
  - Superposition coding of M''_j: $R'' < I(X_1; Y_3|U, X_2) - \delta(\epsilon)$

Partial decode–forward

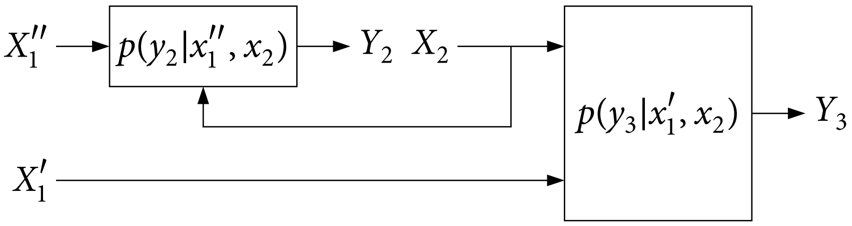
- Partial decode–forward lower bound:

$$C \geq \max_{p(u,x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ I(U; Y_2|X_2) + I(X_1; Y_3|X_2, U)\}$$

- Tight for a semideterministic DM-RC (El Gamal–Aref 1982): $Y_2 = y_2(X_1, X_2)$

$$C = \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3),\ H(Y_2|X_2) + I(X_1; Y_3|X_2, Y_2)\}$$

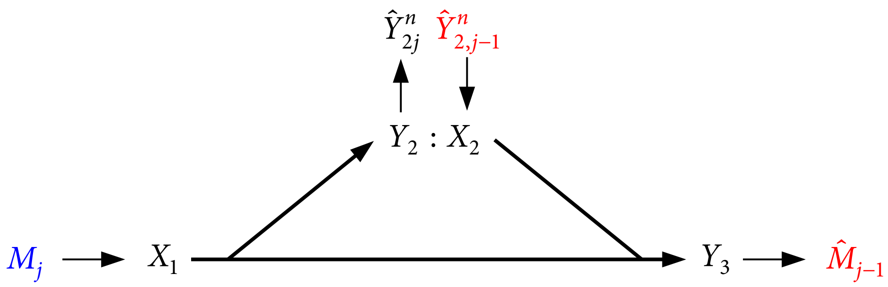
- Tight for a DM-RC with orthogonal sender components (El Gamal–Zahedi 2005)



$$C = \max_{p(x_2)p(x_1'|x_2)p(x_1''|x_2)} \min\{I(X_1', X_2; Y_3),\ I(X_1''; Y_2|X_2) + I(X_1'; Y_3|X_2)\}$$

Compress–forward (Cover–El Gamal 1979)

- Key idea: Compress (quantize) Y_2 instead of fully or partially decoding it



Compress–forward lower bound (El Gamal–Mohseni–Zahedi 2006)

$$C \geq \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2)} \min\{I(X_1, X_2; Y_3) - I(Y_2; \hat{Y}_2|X_1, X_2, Y_3),\ I(X_1; \hat{Y}_2, Y_3|X_2)\}$$

- Tight for some DM-RCs with orthogonal receiver components (read NIT 16.7.3)
  Including example showing that cutset bound is not tight in general

Proof of achievability

- Block Markov coding, compress-bin, simultaneous nonunique decoding
- Codebook generation: Fix $p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2)$ that attains bound. For $j \in [1:b]$:
  - Independently generate 2^{nR} sequences $x_1^n(m_j) \sim \prod_{i=1}^n p_{X_1}(x_{1i})$, $m_j \in [1:2^{nR}]$
  - Independently generate 2^{nR_2} sequences $x_2^n(l_{j-1}) \sim \prod_{i=1}^n p_{X_2}(x_{2i})$, $l_{j-1} \in [1:2^{nR_2}]$
  - For each $l_{j-1} \in [1:2^{nR_2}]$, randomly and conditionally independently generate $2^{n\tilde{R}_2}$ ($\tilde{R}_2 \ge R_2$) sequences $\hat{y}_2^n(k_j|l_{j-1}) \sim \prod_{i=1}^n p_{\hat{Y}_2|X_2}(\hat{y}_{2i}|x_{2i}(l_{j-1}))$, $k_j \in [1:2^{n\tilde{R}_2}]$
- Codebooks: For $j \in [1:b]$,

$$\mathcal{C}_j = \{(x_1^n(m_j), x_2^n(l_{j-1}), \hat{y}_2^n(k_j|l_{j-1})) : m_j \in [1:2^{nR}],\ l_{j-1} \in [1:2^{nR_2}],\ k_j \in [1:2^{n\tilde{R}_2}]\}$$

- Partition the set $[1:2^{n\tilde{R}_2}]$ into 2^{nR_2} equal-size bins $\mathcal{B}(l_j)$, $l_j \in [1:2^{nR_2}]$

Proof of achievability

Block	1	2	3	…	$b-1$	b
X_1	$x_1^n(m_1)$	$x_1^n(m_2)$	$x_1^n(m_3)$	…	$x_1^n(m_{b-1})$	$x_1^n(1)$
Y_2	$\hat{y}_2^n(k_1|1), l_1$	$\hat{y}_2^n(k_2|l_1), l_2$	$\hat{y}_2^n(k_3|l_2), l_3$	…	$\hat{y}_2^n(k_{b-1}|l_{b-2}), l_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(l_1)$	$x_2^n(l_2)$	…	$x_2^n(l_{b-2})$	$x_2^n(l_{b-1})$
Y_3						

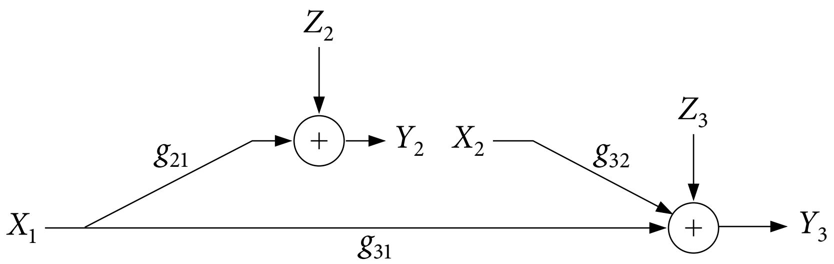
- Encoding:
  - Transmit $x_1^n(m_j)$ from codebook $\mathcal{C}_j$
- Relay encoding:
  - At the end of block j, find k_j such that $(y_2^n(j), \hat{y}_2^n(k_j|l_{j-1}), x_2^n(l_{j-1})) \in \mathcal{T}_{\epsilon'}^{(n)}$ (by covering lemma successful if $\tilde{R}_2 > I(Y_2; \hat{Y}_2|X_2) + \delta(\epsilon')$)
  - In block $j+1$, transmit $x_2^n(l_j)$, where l_j is the bin index of k_j

Proof of achievability

Block	1	2	3	...	$b-1$	b
X_1	$x_1^n(m_1)$	$x_1^n(m_2)$	$x_1^n(m_3)$	...	$x_1^n(m_{b-1})$	$x_1^n(1)$
Y_2	$\hat{y}_2^n(k_1\|1), l_1$	$\hat{y}_2^n(k_2\|l_1), l_2$	$\hat{y}_2^n(k_3\|l_2), l_3$	...	$\hat{y}_2^n(k_{b-1}\|l_{b-2}), l_{b-1}$	$\emptyset$
X_2	$x_2^n(1)$	$x_2^n(l_1)$	$x_2^n(l_2)$	...	$x_2^n(l_{b-2})$	$x_2^n(l_{b-1})$
Y_3	$\emptyset$	$\hat{l}_1, \hat{k}_1, \hat{m}_1$	$\hat{l}_2, \hat{k}_2, \hat{m}_2$	...	$\hat{l}_{b-2}, \hat{k}_{b-2}, \hat{m}_{b-2}$	$\hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}$

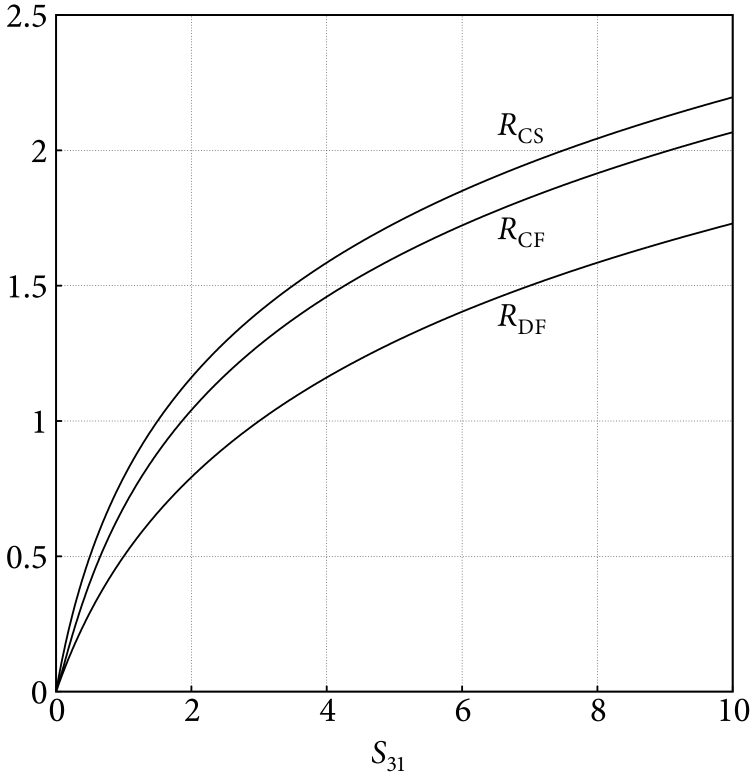
- Decoding:
  - At the end of block $j+1$, find the unique $\hat{l}_j$ such that $(x_2^n(\hat{l}_j), y_3^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$ (successful if $R_2 < I(X_2; Y_3) - \delta(\epsilon)$)
  - Find the unique $\hat{m}_j$ such that $(x_1^n(\hat{m}_j), x_2^n(\hat{l}_{j-1}), \hat{y}_2^n(\hat{k}_j|\hat{l}_{j-1}), y_3^n(j)) \in \mathcal{T}_\epsilon^{(n)}$ for some $\hat{k}_j \in \mathcal{B}(\hat{l}_j)$ (successful if $R + \tilde{R}_2 - R_2 < I(X_1; Y_3|X_2) + I(\hat{Y}_2; X_1, Y_3|X_2) - \delta(\epsilon)$)
- Using F–M to eliminate $R_2, \tilde{R}_2$ yields the two bounds

Gaussian relay channel



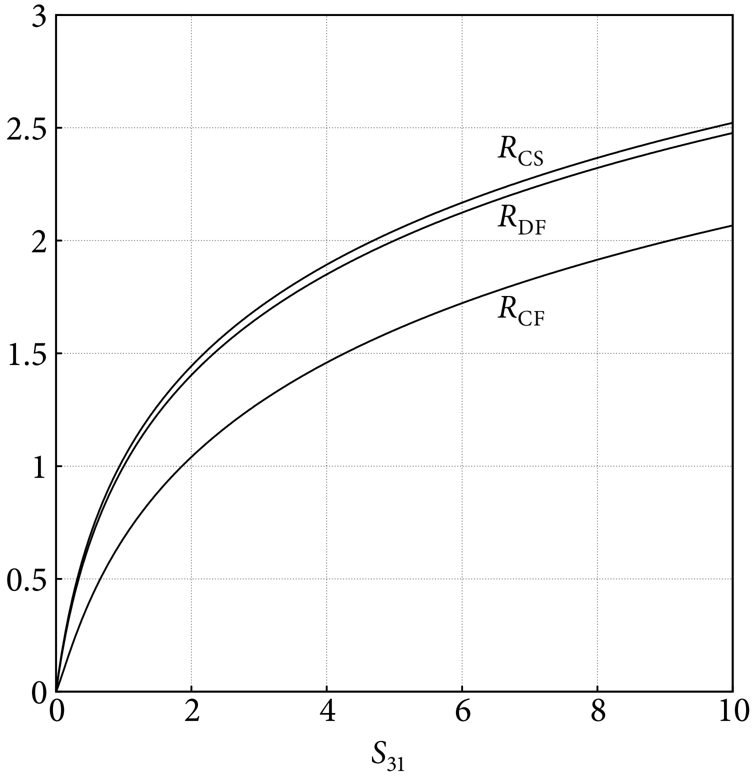
- $Z_2, Z_3 \sim \mathrm{N}(0,1)$ independent of each other
- Power constraints P on X_1 and on X_2
- SNRs: $S_{21} = g_{21}^2 P$, $S_{31} = g_{31}^2 P$, $S_{32} = g_{32}^2 P$
- Capacity not known for any $S_{21}, S_{31}, S_{32} > 0$
- Decode–forward and compress–forward achieve within 1/2 bits of cutset bound
- Partial decode–forward $= \max\{$decode–forward, direct transmission$\}$

Comparison of coding schemes



$S_{21} = S_{31}, S_{32} = 4S_{31}S_{21} = 4S_{31}, S_{32} = S_{31}$

Comparison of coding schemes



$S_{21} = S_{31}, S_{32} = 4S_{31}S_{21} = 4S_{31}, S_{32} = S_{31}$

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Summary

- Discrete memoryless relay channel (DM-RC)
- Cutset bound for the relay channel:
  - Cooperative MAC and BC bounds
  - Not tight in general
- Block Markov coding
- Use of multiple independent codebooks
- Decode–forward:
  - Backward decoding
  - Optimal for the degraded relay channel
- Partial decode–forward
- Compress–forward
- Gaussian relay channel

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