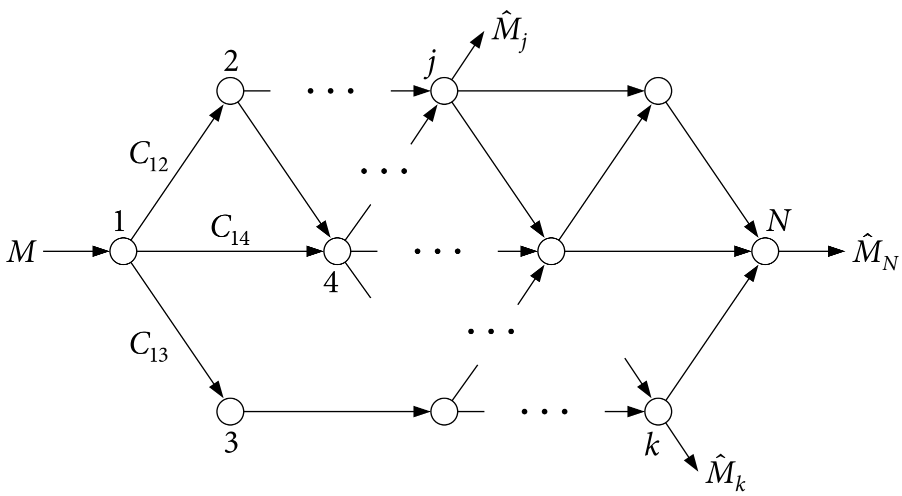
Lecture #9 Graphical Networks

(Reading: NIT 15.1–15.4)

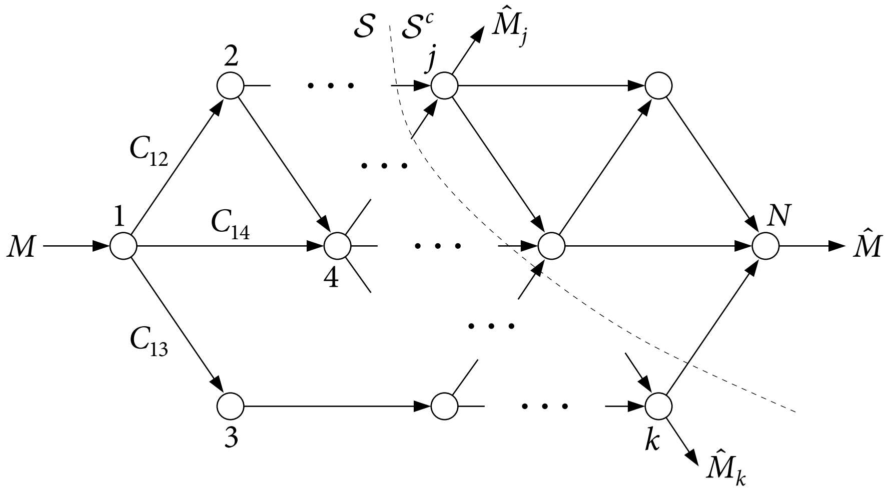
- Graphical multicast network
- Max-flow min-cut theorem
- Network coding theorem
- Graphical multimessage network

Graphical multicast network



- Weighted directed acyclic graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{C})$
- A $(2^{nR}, n)$ code:
  - A message set $[1:2^{nR}]$
  - Source encoder $m_{1j}(m) \in [1:2^{nC_{1j}}]$
  - Relay encoder $k \in [2:N]$: $m_{kl}(m_{jk}\colon (j,k) \in \mathcal{E}) \in [1:2^{nC_{kl}}]$
  - Decoder $k \in \mathcal{D}$: $\hat{m}_k(m_{jk}\colon (j,k) \in \mathcal{E})$
- $P_e^{(n)}$, achievability, and C: Defined as before

Cutset upper bound on C



Theorem 15.1

$$C \le \min_{j \in \mathcal{D}} \min_{\substack{\mathcal{S} \subset \mathcal{N} \\ 1 \in \mathcal{S},\, j \in \mathcal{S}^c}} C(\mathcal{S}),$$

where $C(\mathcal{S}) = \sum_{k \in \mathcal{S},\, l \in \mathcal{S}^c} C_{kl}$

Proof

- Consider a cut $(\mathcal{S}, \mathcal{S}^c)$ and a destination node $j \in \mathcal{S}^c$
- For any $(2^{nR}, n)$ code, $\hat{M}_j$ is a function of $M(\mathcal{S}, \mathcal{S}^c) = \{M_{kl} : k \in \mathcal{S},\, l \in \mathcal{S}^c\}$
- By Fano's inequality,

$$\begin{aligned} nR &\le I(M; \hat{M}_j) + n\epsilon_n \\ &\le H(\hat{M}_j) + n\epsilon_n \\ &\le H(M(\mathcal{S}, \mathcal{S}^c)) + n\epsilon_n \\ &\le nC(\mathcal{S}) + n\epsilon_n \end{aligned}$$

- Since the above holds for every cut and every destination, in the limit

$$R \le \min_{j \in \mathcal{D}} \min_{\substack{\mathcal{S} \subset \mathcal{N} \\ 1 \in \mathcal{S},\, j \in \mathcal{S}^c}} C(\mathcal{S})$$

Capacity of graphical unicast network

- Unicast: $\mathcal{D} = \{N\}$

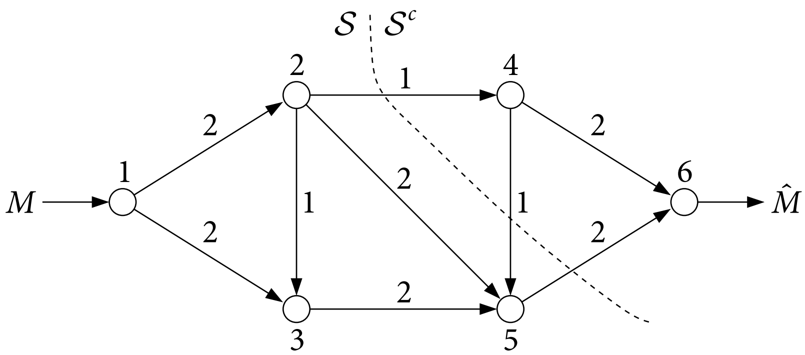
Theorem 15.2 (Max-flow min-cut theorem (Ford–Fulkerson 1956))

$$C = \min_{\substack{\mathcal{S} \subset \mathcal{N} \\ 1 \in \mathcal{S}, N \in \mathcal{S}^c}} C(\mathcal{S})$$

- Capacity is achieved error-free using routing with finite block length (NIT 15.2)
- Information can be treated as a commodity
- Ford–Fulkerson algorithm finds capacity and optimal routing
- Continues to hold for networks with cycles and delays

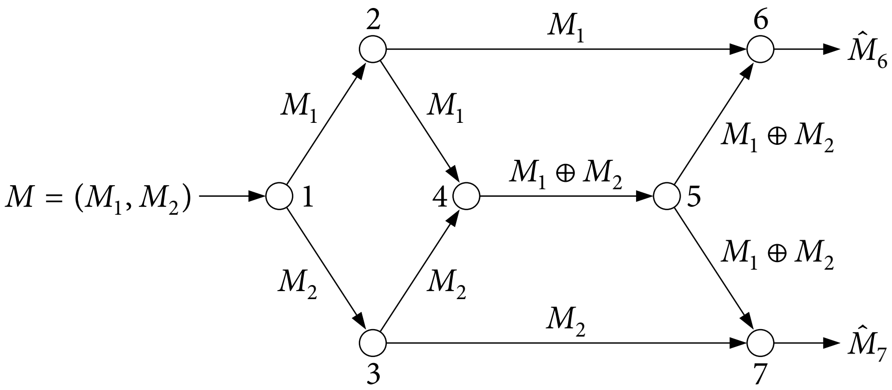
Capacity of graphical unicast network

- Example:



  - $C = 3$
  - Minimum cut: $\mathcal{S} = \{1, 2, 3, 5\}$
  - Achieved by routing 1 bit along $1 \to 2 \to 4 \to 6$ and 2 bits along $1 \to 3 \to 5 \to 6$
- Cutset bound also achievable for multicast networks:
  - Not always achieved using only routing

Butterfly multicast network



- Cutset bound: $C \leq 2$
- Routing achieves $R = R_1 + R_2 = 1$
- Achieving cutset bound requires network coding!

  Treating information as a commodity is not optimal in general

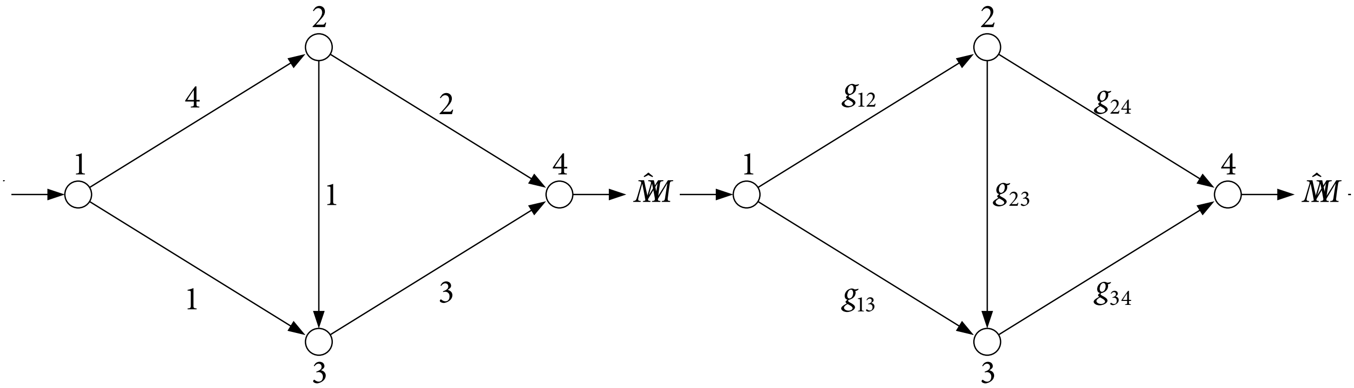
Capacity of graphical multicast network

Network coding theorem (Ahlswede–Cai–Li–Yeung 2000)

$$C = \min_{j \in \mathcal{D}} \min_{\substack{\mathcal{S} \subset \mathcal{N} \\ 1 \in \mathcal{S}, j \in \mathcal{S}^c}} C(\mathcal{S})$$

- Achieved by random binning (Ahlswede–Cai–Li–Yeung 2000)
- Also achieved error free using linear coding: Li–Yeung–Cai (2003), Koetter–Médard (2003) (NIT 15.3.1)
- Continues to hold for networks with broadcasting, cycles, and delays

Proof of achievability via binning: Unicast case



- Random codebook generation:
  - Source encoder: Generate $g_{1j}(m) \sim \mathrm{Unif}[1:2^{nC_{1j}}]$ for each $m \in [1:2^{nR}]$, e.g.,

    $$g_{12}(m) \sim \mathrm{Unif}[1:2^{4n}] \quad \text{and} \quad g_{13}(m) \sim \mathrm{Unif}[1:2^{n}]$$
  - Relay encoder: Generate $g_{kl}(m_{*k}) \sim \mathrm{Unif}[1:2^{nC_{kl}}]$ for each $m_{*k} = (m_{jk}: (j,k) \in \mathcal{E})$, e.g.,

    $$g_{34}(m_{13}, m_{23}) \sim \mathrm{Unif}[1:2^{3n}]$$
  - The mappings g_{jk}, $(j,k) \in \mathcal{E}$, induce functions of m at every edge, e.g., $m_{34}(m) = g_{34}(g_{13}(m), g_{23}(g_{12}(m)))$
- Encoding: To send message m, node j transmits $m_{jk}(m)$ over edge $(j,k) \in \mathcal{E}$

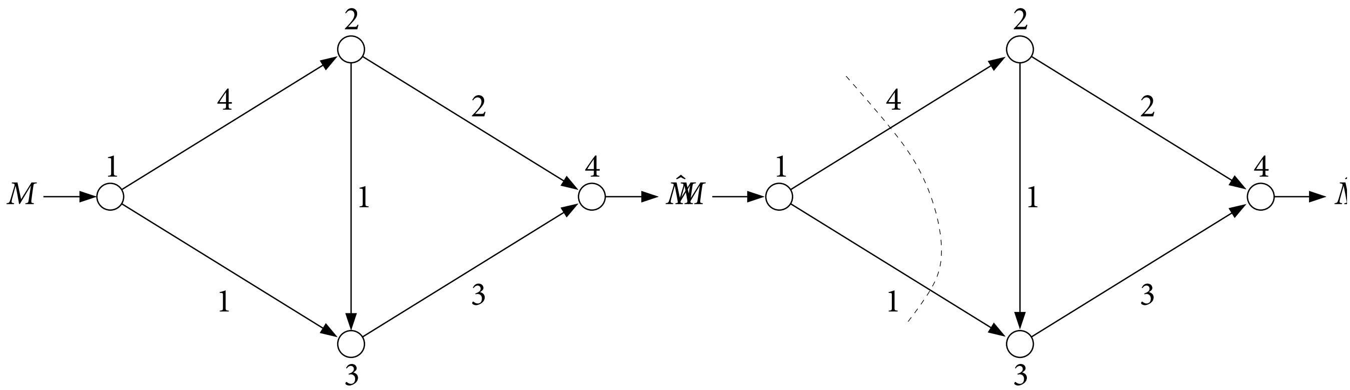
Analysis of the probability of error

- Consider the probability of error averaged over M and $(G_{jk}: (j,k) \in \mathcal{E})$:

$$\begin{aligned}
\mathrm{P}(\mathcal{E}) &= \mathrm{P}\{M \neq \hat{M}\} \\
&= \frac{1}{2^{nR}} \sum_{m} \mathrm{P}\left\{M_{*N}(m) = M_{*N}(\tilde{m}) \text{ for some } \tilde{m} \neq m | M = m\right\} \\
&= \mathrm{P}\left\{M_{*N}(1) = M_{*N}(\tilde{m}) \text{ for some } \tilde{m} \neq 1\right\} \\
&\leq \sum_{\tilde{m} \neq 1} \mathrm{P}\left\{M_{*N}(1) = M_{*N}(\tilde{m})\right\} \\
&= (2^{nR} - 1)\, \mathrm{P}\left\{M_{*N}(1) = M_{*N}(2)\right\}
\end{aligned}$$

- Thus, it suffices to bound the collision probability $\mathrm{P}\{M_{*N}(1) = M_{*N}(2)\}$

Collision probability



- We decompose $\mathsf{P}\{M_{*N}(1) = M_{*N}(2)\}$ by different cuts ($N = 4$):

$$
\begin{aligned}
\mathsf{P}\left\{M_{*4}(1) = M_{*4}(2)\right\} = \mathsf{P}\left\{M_{*4}(1) = M_{*4}(2),\ M_{*2}(1) = M_{*2}(2), M_{*3}(1) = M_{*3}(2)\right\} \\
+ \mathsf{P}\left\{M_{*4}(1) = M_{*4}(2),\ M_{*2}(1) \neq M_{*2}(2), M_{*3}(1) = M_{*3}(2)\right\} \\
+ \mathsf{P}\left\{M_{*4}(1) = M_{*4}(2),\ M_{*2}(1) = M_{*2}(2), M_{*3}(1) \neq M_{*3}(2)\right\} \\
+ \mathsf{P}\left\{M_{*4}(1) = M_{*4}(2),\ M_{*2}(1) \neq M_{*2}(2), M_{*3}(1) \neq M_{*3}(2)\right\}
\end{aligned}
$$

- No collision at node 1, collision at nodes $2, 3, 4$:

$$\mathsf{P}\left\{M_{*4}(1) = M_{*4}(2),\ M_{*2}(1) = M_{*2}(2), M_{*3}(1) = M_{*3}(2)\right\}$$

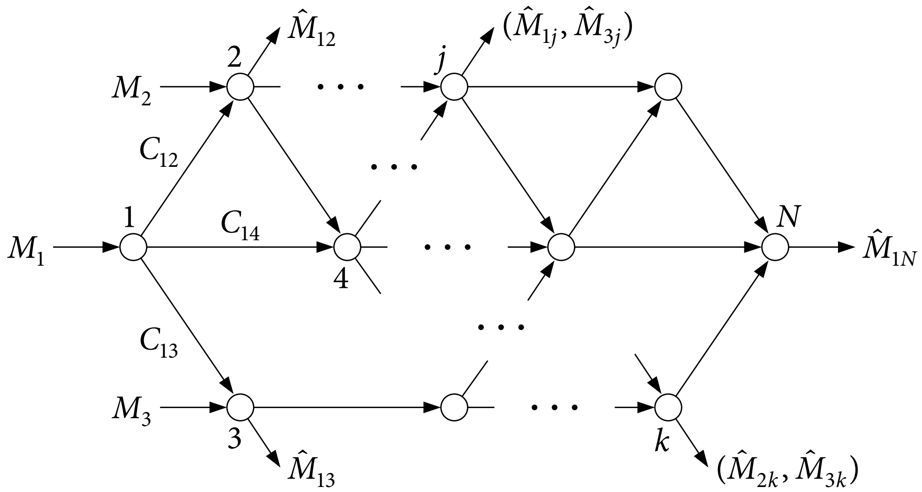
Proof of achievability: Multicast case

- We repeat the unicast argument for each destination $j \in \mathcal{D}$
- By the union of events bound,

$$
\begin{aligned}
\mathsf{P}(\mathcal{E}) &\leq \sum_{j \in \mathcal{D}} 2^{N-2} \frac{1}{2^{n(\min_{\mathcal{S}:1\in\mathcal{S}, j\in\mathcal{S}^c} C(\mathcal{S}) - R)}} \\
&\leq |\mathcal{D}| 2^{N-2} \frac{1}{2^{n(\min_{j\in\mathcal{D}} \min_{\mathcal{S}:1\in\mathcal{S}, j\in\mathcal{S}^c} C(\mathcal{S}) - R)}},
\end{aligned}
$$

which tends to 0 as $n \to \infty$, if $R < \min_{j\in\mathcal{D}} \min_{\mathcal{S}:1\in\mathcal{S}, j\in\mathcal{S}^c} C(\mathcal{S})$

Graphical multimessage network



- Each node $j \in [1:N-1]$ wishes to send M_j to a set $\mathcal{D}_j \subseteq [j+1:N]$
- A $(2^{nR_1}, \ldots, 2^{nR_{N-1}}, n)$ code:
  - Message sets $[1:2^{nR_1}], \ldots, [1:2^{nR_{N-1}}]$
  - Encoder $k \in [1:N-1]$: $m_{kl}(m_k, m_{jk}: (j,k) \in \mathcal{E})$
  - Decoder $l \in \cup_j \mathcal{D}_j$: $\hat{m}_{jl}(m_{kl}: (k,l) \in \mathcal{E})$ for j such that $l \in \mathcal{D}_j$
- $P_e^{(n)}$, achievability, and $\mathscr{C}$: defined as before

Cutset bound for graphical multimessage network

Theorem 15.4

If $(R_1, \ldots, R_{N-1})$ is achievable, then

$$\sum_{j \in \mathcal{S}} R_j \leq C(\mathcal{S})$$

for all $\mathcal{S} \subset \mathcal{N}$ such that $\mathcal{D}_j \cap \mathcal{S}^c \neq \emptyset$ for some $j \in [1:N-1]$

- Tight for multimessage multicast ($\mathcal{D}_j = \mathcal{D}, j \in [1:N-1]$)
- Will be later extended to noisy networks
- Not tight in general

Multimessage multicast

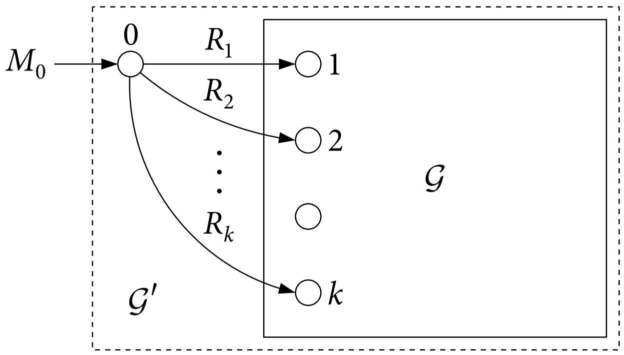
Theorem 15.5 (Ahlswede–Cai–Li–Yeung 2000)

The capacity region is the set of $(R_1, \ldots, R_k)$ such that

$$\sum_{j \in \mathcal{S}} R_j \leq C(\mathcal{S})$$

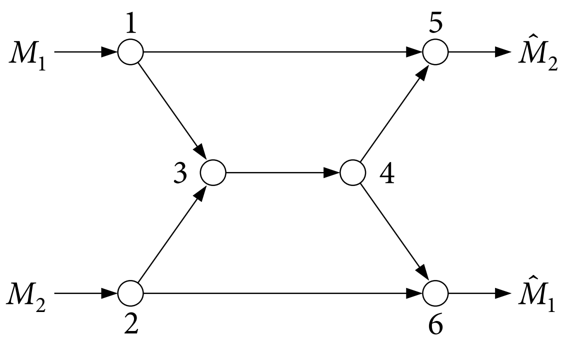
for all $\mathcal{S} \subset \mathcal{N}$ with $[1:k] \cap \mathcal{S} \neq \emptyset$ and $\mathcal{D} \cap \mathcal{S}^c \neq \emptyset$

- Key idea: augmented single-message multicast network (Read NIT 15.4.1)

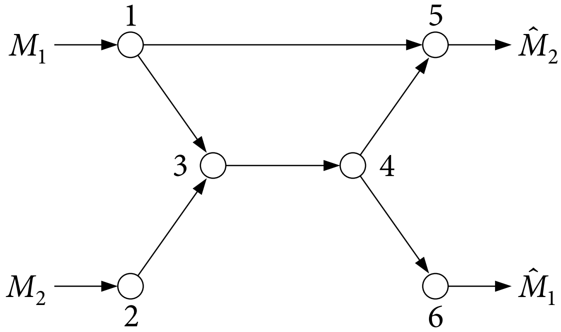


Multiple unicast

- Routing is insufficient: Consider the following example ($C_{jk} = 1$)



- Cutset bound is not tight: Consider the following example ($C_{jk} = 1$)



- Capacity is not known in general

Summary

- Cutset bounds on the capacity of graphical networks
- Max-flow min-cut theorem for graphical unicast networks
- Routing alone does not achieve the capacity of general graphical networks
- Network coding theorem for graphical multicast networks
- Linear network coding achieves the capacity of graphical multimessage multicast networks (error-free and with finite block length)

References

Ahlswede, R., Cai, N., Li, S.-Y. R., and Yeung, R. W. (2000). Network information flow. *IEEE Trans. Inf. Theory*, 46(4), 1204–1216.

Ford, L. R., Jr. and Fulkerson, D. R. (1956). Maximal flow through a network. *Canad. J. Math.*, 8(3), 399–404.

Koetter, R. and Médard, M. (2003). An algebraic approach to network coding. *IEEE/ACM Trans. Netw.*, 11(5), 782–795.

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