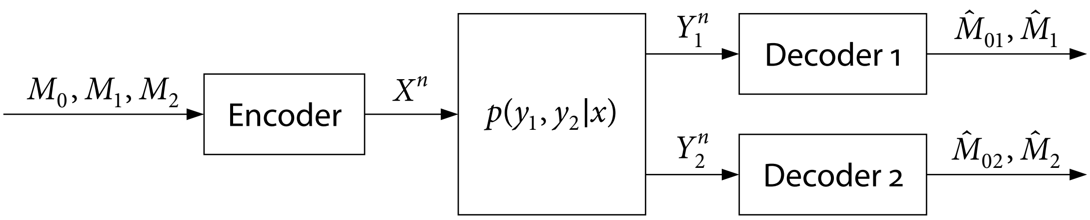
Lecture #8 General broadcast channels

(Reading: NIT 8.3, 9.6)

- Marton's inner bound
- Gaussian vector broadcast channel
- Marton's inner bound with common message
- Nair–El Gamal outer bound

Broadcast communication system



- DM broadcast channel (BC) $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1 \times \mathcal{Y}_2)$
- $(2^{nR_0}, 2^{nR_1}, 2^{nR_2}, n)$ code, $P_e^{(n)}$, achievability, $\mathscr{C}$: Same as before
- Superposition coding is optimal for
  - Degraded BC: $X \to Y_1 \to Y_2$
  - Less noisy BC: $I(U; Y_1) \geq I(U; Y_2)$ for every $p(u, x)$
  - More capable BC: $I(X; Y_1) \geq I(X; Y_2)$ for every $p(x)$
  - General BC with degraded message sets ($R_1 = 0$ or $R_2 = 0$): Read NIT 8.1
- This lecture: Inner bound on private-message capacity region of general BC

Marton's inner bound

- A simple inner bound: (R_1, R_2) is achievable for the DM-BC $p(y_1, y_2|x)$ if

$$R_1 < I(U_1; Y_1),$$
$$R_2 < I(U_2; Y_2)$$

  for some pmf $p(u_1)p(u_2)$ and function $x(u_1, u_2)$

- Marton's coding scheme: Allows U_1 and U_2 to be dependent

Theorem 8.3 (Marton 1979)

(R_1, R_2) is achievable if

$$R_1 < I(U_1; Y_1),$$
$$R_2 < I(U_2; Y_2),$$
$$R_1 + R_2 < I(U_1; Y_1) + I(U_2; Y_2) - I(U_1; U_2)$$

for some pmf $p(u_1, u_2)$ and function $x(u_1, u_2)$

- Region is not convex in general; can be convexified via Q

Semideterministic BC

- Marton inner bound tight for semideterministic BC ($Y_1 = y_1(X)$): Set $U_1 = Y_1$
  The capacity region is the set of (R_1, R_2) such that

$$R_1 \leq H(Y_1),$$
$$R_2 \leq I(U; Y_2),$$
$$R_1 + R_2 \leq H(Y_1|U) + I(U; Y_2)$$

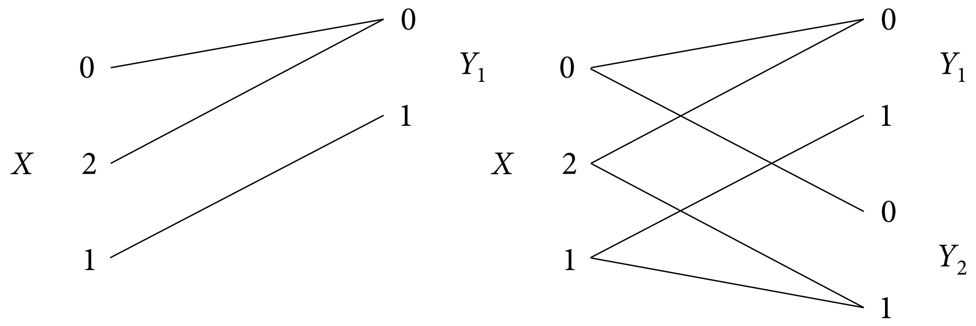
  for some $p(u, x)$

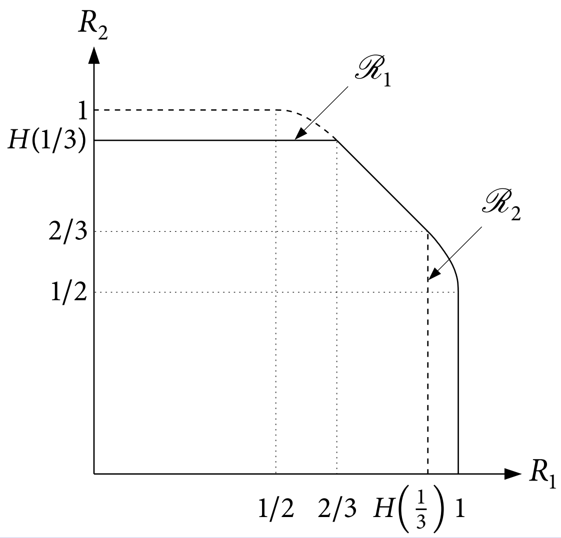
- Deterministic BC ($Y_1 = y_1(X), Y_2 = y_2(X)$): Further set $U_2 = Y_2$
  The capacity region is the set of (R_1, R_2) such that

$$R_1 \leq H(Y_1),$$
$$R_2 \leq H(Y_2),$$
$$R_1 + R_2 \leq H(Y_1, Y_2)$$

  for some $p(x)$

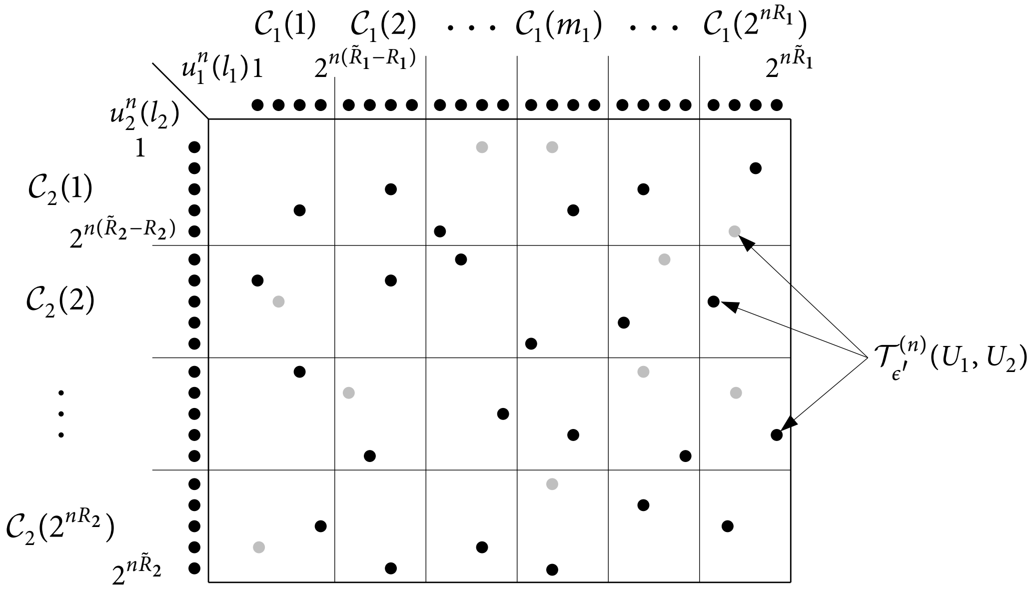
Example: Blackwell channel





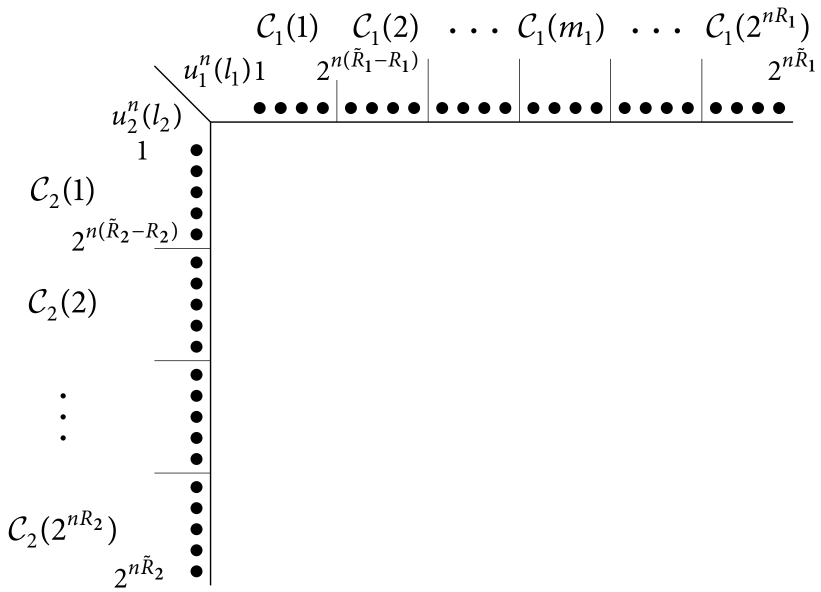
Proof of achievability

- Use multicoding and the mutual covering lemma



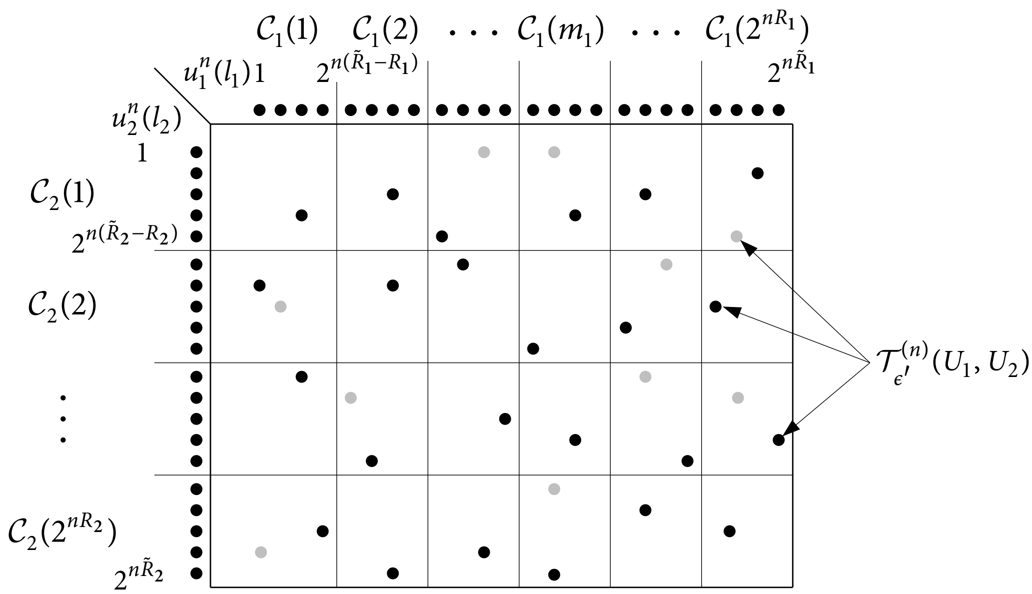
Proof of achievability

- Codebook generation: Fix $p(u_1, u_2)$ and $x(u_1, u_2)$
  - For each $m_1 \in [1 : 2^{nR_1}]$ generate a subcodebook $\mathcal{C}_1(m_1)$ consisting of $2^{n(\tilde{R}_1 - R_1)}$ sequences $u_1^n(l_1) \sim \prod_{i=1}^n p_{U_1}(u_{1i})$, $l_1 \in [(m_1 - 1)2^{n(\tilde{R}_1 - R_1)} + 1 : m_1 2^{n(\tilde{R}_1 - R_1)}]$
  - Similarly, generate $\mathcal{C}_2(m_2)$, $m_2 \in [1 : 2^{nR_2}]$



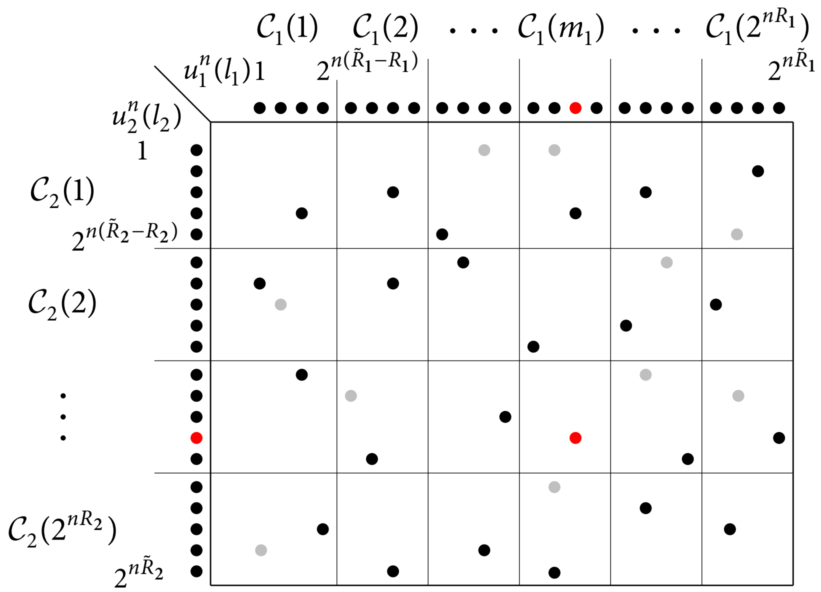
Proof of achievability

- Codebook generation: Fix $p(u_1, u_2)$ and $x(u_1, u_2)$
  - For each (m_1, m_2), find (l_1, l_2) such that $u_1^n(l_1) \in \mathcal{C}_1(m_1)$, $u_2^n(l_2) \in \mathcal{C}_2(m_2)$, $(u_1^n(l_1), u_2^n(l_2)) \in \mathcal{T}_{\epsilon'}^{(n)}$
  - If no such pair exists, choose $(l_1, l_2) = (1, 1)$
  - Generate $x^n(m_1, m_2)$ as $x_i(m_1, m_2) = x(u_{1i}(l_1), u_{2i}(l_2))$, $i \in [1 : n]$



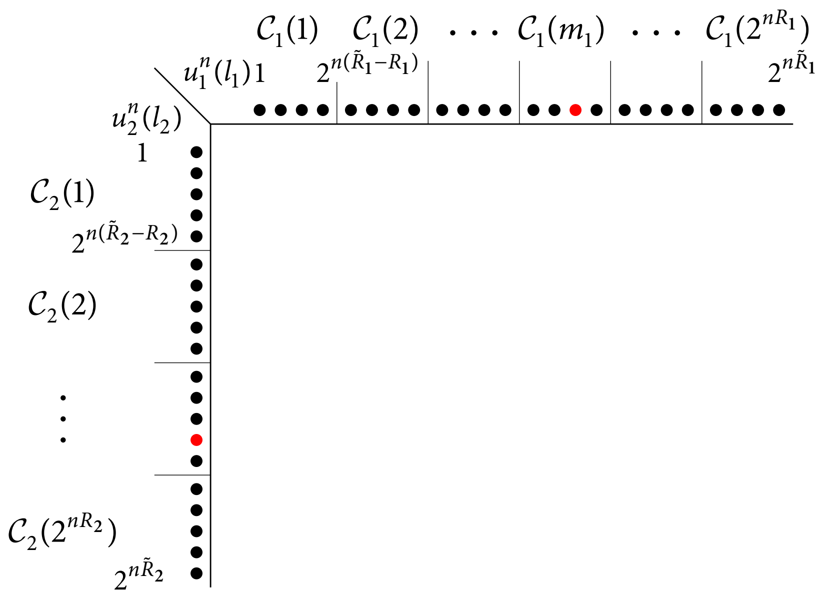
Proof of achievability

- Encoding:
  - To send a message pair (m_1, m_2), transmit $x^n(m_1, m_2)$



Proof of achievability

- Decoding:
  - Decoder $j = 1, 2$ finds unique $\hat{m}_j$ such that $(u_j^n(l_j), y_j^n) \in \mathcal{T}_\epsilon^{(n)}$ for some $u_j^n(l_j) \in \mathcal{C}_j(\hat{m}_j)$



Analysis of the probability of error

- Consider $\mathsf{P}(\mathcal{E})$ conditioned on $(M_1, M_2) = (1, 1)$
- Let (L_1, L_2) denote the pair of chosen indices
- Error events for decoder 1:

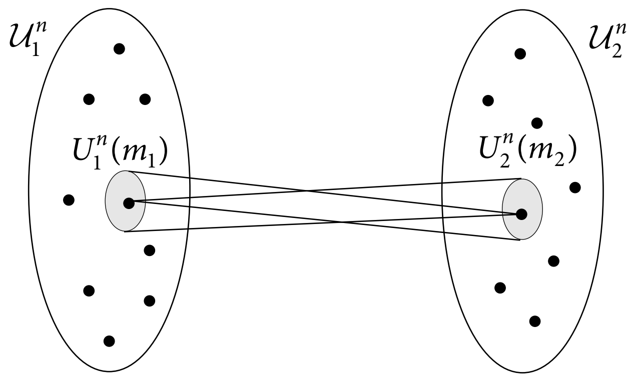
$$\mathcal{E}_0 = \{(U_1^n(l_1), U_2^n(l_2)) \notin \mathcal{T}_{\epsilon'}^{(n)} \text{ for all } (U_1^n(l_1), U_2^n(l_2)) \in \mathcal{C}_1(1) \times \mathcal{C}_2(1)\},$$
$$\mathcal{E}_{11} = \{(U_1^n(L_1), Y_1^n) \notin \mathcal{T}_{\epsilon}^{(n)}\},$$
$$\mathcal{E}_{12} = \{(U_1^n(l_1), Y_1^n) \in \mathcal{T}_{\epsilon}^{(n)}(U_1, Y_1) \text{ for some } l_1 \notin [1 : 2^{n(\tilde{R}_1 - R_1)}]\}$$

Thus, by the union of events bound,

$$\mathsf{P}(\mathcal{E}_1) \le \mathsf{P}(\mathcal{E}_0) + \mathsf{P}(\mathcal{E}_0^c \cap \mathcal{E}_{11}) + \mathsf{P}(\mathcal{E}_{12})$$

Mutual covering lemma ($U_0 = \emptyset$)

- Let $(U_1, U_2) \sim p(u_1, u_2)$ and $\epsilon' < \epsilon$
- For $j = 1, 2$, let $U_j^n(m_j) \sim \prod_{i=1}^n p_{U_j}(u_{ji})$, $m_j \in [1 : 2^{nr_j}]$, be pairwise independent
- Assume that $\{U_1^n(m_1)\}$ and $\{U_2^n(m_2)\}$ are independent



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- Assume that $\{U_1^n(m_1)\}$ and $\{U_2^n(m_2)\}$ are independent

Lemma 8.1 (Mutual covering lemma)

There exists $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ such that

$$\lim_{n\to\infty} \mathsf{P}\{(U_1^n(m_1), U_2^n(m_2)) \notin \mathcal{T}_\epsilon^{(n)} \text{ for all } m_1 \in [1:2^{nr_1}], m_2 \in [1:2^{nr_2}]\} = 0$$

if $r_1 + r_2 > I(U_1; U_2) + \delta(\epsilon)$

- Proof: See NIT Appendix 8A
- This lemma extends the covering lemma:
  - For a single U_1^n sequence ($r_1 = 0$), $r_2 > I(U_1; U_2) + \delta(\epsilon)$ as in the covering lemma
  - Pairwise independence: linear codes for finite field models

Analysis of the probability of error

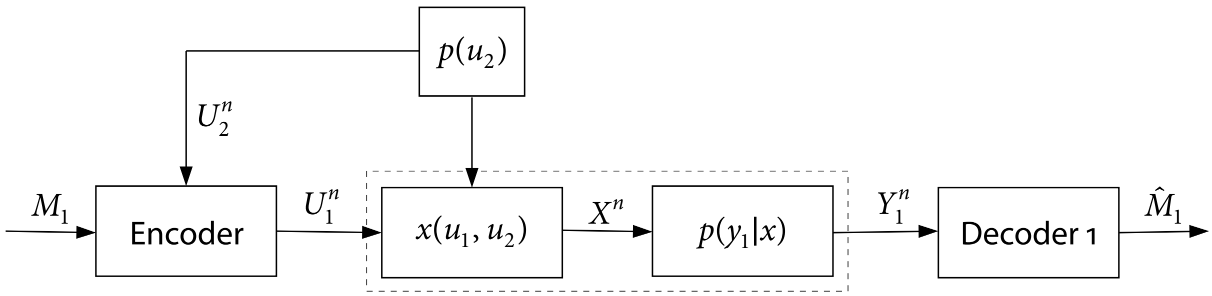
- Error events for decoder 1:

$$\mathcal{E}_0 = \{(U_1^n(l_1), U_2^n(l_2)) \notin \mathcal{T}_{\epsilon'}^{(n)} \text{ for all } (U_1^n(l_1), U_2^n(l_2)) \in \mathcal{C}_1(1) \times \mathcal{C}_2(1)\},$$
$$\mathcal{E}_{11} = \{(U_1^n(L_1), Y_1^n) \notin \mathcal{T}_\epsilon^{(n)}\},$$
$$\mathcal{E}_{12} = \{(U_1^n(l_1), Y_1^n) \in \mathcal{T}_\epsilon^{(n)}(U_1, Y_1) \text{ for some } l_1 \notin [1:2^{n(\tilde{R}_1 - R_1)}]\}$$

- By the mutual covering lemma (with $r_1 = \tilde{R}_1 - R_1$ and $r_2 = \tilde{R}_2 - R_2$), $\mathsf{P}(\mathcal{E}_0) \to 0$ if $(\tilde{R}_1 - R_1) + (\tilde{R}_2 - R_2) > I(U_1; U_2) + \delta(\epsilon')$
- Since $\mathcal{E}_0^c = \{(U_1^n(L_1), U_2^n(L_2), X^n) \in \mathcal{T}_{\epsilon'}^{(n)}\}$, by the conditional typicality lemma, $\mathsf{P}(\mathcal{E}_0^c \cap \mathcal{E}_{11}) \to 0$
- By the packing lemma, $\mathsf{P}(\mathcal{E}_{12}) \to 0$ if $\tilde{R}_1 < I(U_1; Y_1) - \delta(\epsilon)$
- Similarly, $\mathsf{P}(\mathcal{E}_2) \to 0$ if $\tilde{R}_2 < I(U_2; Y_2) + \delta(\epsilon)$
- Using Fourier–Motzkin to eliminate $\tilde{R}_1$ and $\tilde{R}_2$ completes the proof

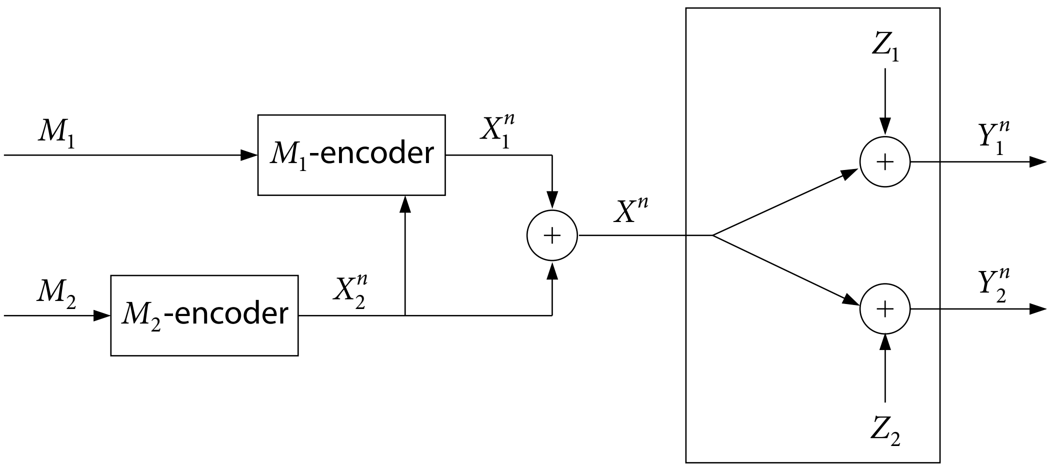
Relationship to Gelfand–Pinsker

- Consider the Marton coding scheme
- Fix $p(u_1, u_2)$ and $x(u_1, u_2)$. This defines a pentagon region
- Consider corner point $(R_1 = I(U_1; Y_1) - I(U_1; U_2), R_2 = I(U_2; Y_2))$
- Marton scheme for communicating M_1 is equivalent to G–P for $p(y_1|u_1, u_2)p(u_2)$



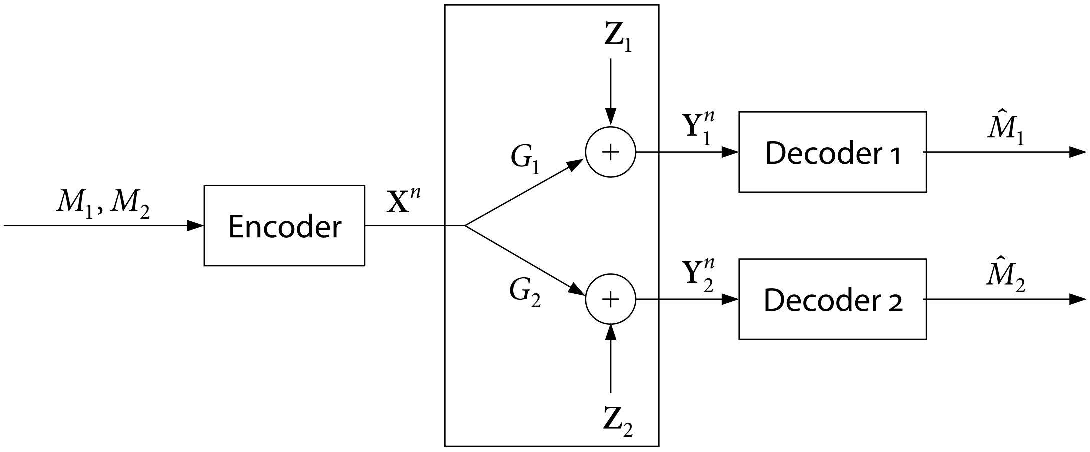
- The corner point $(R_1 = I(U_1; Y_1), R_2 = I(U_2; Y_2) - I(U_1; U_2))$ achieved similarly
- The rest of the pentagon region is achieved by time sharing

Application: Gaussian BC



- Decompose X into the sum of independent $X_1 \sim \mathrm{N}(0, \alpha P)$ and $X_2 \sim \mathrm{N}(0, \bar{\alpha} P)$
- Send M_2 to $Y_2 = X_2 + X_1 + Z_2$: $R_2 < \mathrm{C}(\bar{\alpha}P/(\alpha P + N_2))$ (treat X_1 as noise)
- Send M_1 to $Y_1 = X_1 + X_2 + Z_1$: $R_1 < \mathrm{C}(\alpha P/N_1)$ (writing on dirty paper)
  - Substitute $U_2 = X_2$ and $U_1 = \beta U_2 + X_1$, $\beta = \alpha P/(\alpha P + N_1)$ in Marton
- This coding scheme works even when $N_1 > N_2$ (unlike superposition coding)

Gaussian vector broadcast channel



- $\mathbf{Z}_1, \mathbf{Z}_2 \sim \mathrm{N}(0, I_r)$
- Average power constraint: $\sum_{i=1}^{n} \mathbf{x}^T(m_1, m_2, i)\mathbf{x}(m_1, m_2, i) \le nP$
- Channel is not degraded in general (superposition coding not optimal)
- Marton coding (vector writing on dirty paper) is optimal, however

Capacity region

- $\mathscr{R}_1$: (R_1, R_2) such that

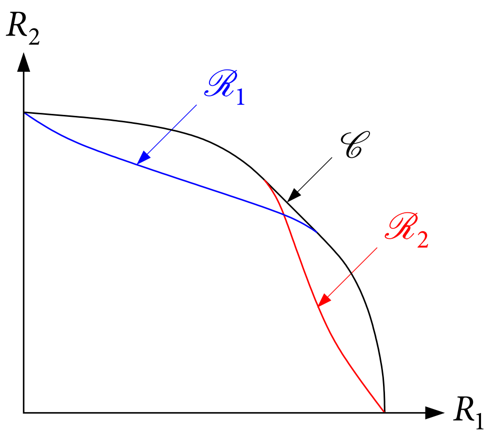
$$R_1 < \frac{1}{2}\log\frac{|G_1K_1G_1^T + G_1K_2G_1^T + I_r|}{|G_1K_2G_1^T + I_r|},$$
$$R_2 < \frac{1}{2}\log|G_2K_2G_2^T + I_r|$$

  for some $K_1, K_2 \succeq 0$ with $\operatorname{tr}(K_1 + K_2) \le P$

- $\mathscr{R}_2$: (R_1, R_2) such that

$$R_1 < \frac{1}{2}\log|G_1K_1G_1^T + I_r|,$$
$$R_2 < \frac{1}{2}\log\frac{|G_2K_2G_2^T + G_2K_1G_2^T + I_r|}{|G_2K_1G_2^T + I_r|}$$

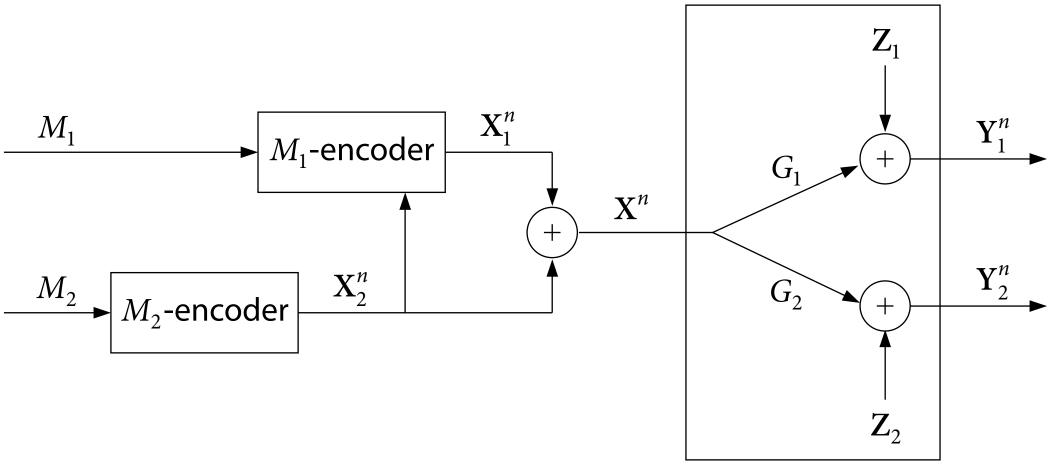
  for some $K_1, K_2 \succeq 0$ with $\operatorname{tr}(K_1 + K_2) \le P$



Theorem 9.4 (Weingarten–Steinberg–Shamai 2006)

$\mathscr{C}$ is the convex closure of $\mathscr{R}_1 \cup \mathscr{R}_2$

Proof of achievability for $\mathscr{R}_2$



- Decompose $\mathbf{X}$ into the sum of independent $\mathbf{X}_1 \sim \mathrm{N}(0, K_1)$ and $\mathbf{X}_2 \sim \mathrm{N}(0, K_2)$
- Send M_2 to $\mathbf{Y}_2 = G_2\mathbf{X}_2 + G_1\mathbf{X}_1 + \mathbf{Z}_2$: $R_2 < \frac{1}{2} \log \frac{|G_2 K_2 G_2^T + G_2 K_1 G_2^T + I_r|}{|G_2 K_1 G_2^T + I_r|}$
- Send M_1 to $\mathbf{Y}_1 = G_1\mathbf{X}_1 + G_2\mathbf{X}_2 + \mathbf{Z}_1$: $R_1 < \frac{1}{2} \log |G_1 K_1 G_1^T + I_r|$ (vector WDP)
- $\mathscr{R}_1$ is achieved similarly

Marton's inner bound with common message

Theorem 8.4 (Marton 1979, Liang 2005)

(R_0, R_1, R_2) is achievable if

$$\begin{aligned}
R_0 + R_1 &< I(U_0, U_1; Y_1), \\
R_0 + R_2 &< I(U_0, U_2; Y_2), \\
R_0 + R_1 + R_2 &< I(U_0, U_1; Y_1) + I(U_2; Y_2 | U_0) - I(U_1; U_2 | U_0), \\
R_0 + R_1 + R_2 &< I(U_1; Y_1 | U_0) + I(U_0, U_2; Y_2) - I(U_1; U_2 | U_0), \\
2R_0 + R_1 + R_2 &< I(U_0, U_1; Y_1) + I(U_0, U_2; Y_2) - I(U_1; U_2 | U_0)
\end{aligned}$$

for some $p(u_0, u_1, u_2)$ and function $x(u_0, u_1, u_2)$

- Proof of achievability: Superposition coding + Marton coding
- Tight for all classes of BCs with known capacity regions
- Even for $R_0 = 0$, larger than Marton's inner bound with $U_0 = \emptyset$ (Theorem 8.3)

Nair–El Gamal outer bound

Theorem 8.6 (Nair–El Gamal 2007)

If (R_0, R_1, R_2) is achievable, then

$$\begin{aligned} R_0 &\le \min\{I(U_0; Y_1),\ I(U_0; Y_2)\}, \\ R_0 + R_1 &\le I(U_0, U_1; Y_1), \\ R_0 + R_2 &\le I(U_0, U_2; Y_2), \\ R_0 + R_1 + R_2 &\le I(U_0, U_1; Y_1) + I(U_2; Y_2 | U_0, U_1), \\ R_0 + R_1 + R_2 &\le I(U_1; Y_1 | U_0, U_2) + I(U_0, U_2; Y_2) \end{aligned}$$

for some $p(u_1)p(u_2)p(u_0|u_1, u_2)$ and function $x(u_0, u_1, u_2)$

- Tight for all BCs with known capacity regions that we discussed so far
- Does not coincide with Marton's inner bound (Jog–Nair 2010)
- Not tight in general (Geng–Gohari–Nair–Yu 2011)

Summary

- Marton's inner bound:
  - Multidimensional subcodebook generation
  - Generating correlated codewords for independent messages
- Mutual covering lemma
- Connection between Marton coding and Gelfand–Pinsker coding
- Writing on dirty paper achieves the capacity region of the Gaussian vector BC
- Nair–El Gamal outer bound

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