Lecture #8  General broadcast channels

(Reading: NIT 8.3, 9.6)

- Marton’s inner bound
- Gaussian vector broadcast channel
- Marton’s inner bound with common message
- Nair–El Gamal outer bound

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Broadcast communication system

- DM broadcast channel (BC) \((\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1 \times \mathcal{Y}_2)\)
- \((2^{nR_0}, 2^{nR_1}, 2^{nR_2}, n)\) code, \(P_e^{(n)}\), achievability, \(\mathcal{C}\): Same as before
- **Superposition coding** is optimal for
  - Degraded BC: \(X \rightarrow Y_1 \rightarrow Y_2\)
  - Less noisy BC: \(I(U; Y_1) \geq I(U; Y_2)\) for every \(p(u, x)\)
  - More capable BC: \(I(X; Y_1) \geq I(X; Y_2)\) for every \(p(x)\)
  - General BC with degraded message sets \((R_1 = 0\) or \(R_2 = 0)\): Read NIT 8.1
- **This lecture**: Inner bound on private-message capacity region of general BC

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Marton’s inner bound

- A simple inner bound: \((R_1, R_2)\) is achievable for the DM-BC \(p(y_1, y_2|x)\) if

\[
R_1 < I(U_1; Y_1), \\
R_2 < I(U_2; Y_2)
\]

for some pmf \(p(u_1)p(u_2)\) and function \(x(u_1, u_2)\)

- Marton’s coding scheme: Allows \(U_1\) and \(U_2\) to be dependent

**Theorem 8.3 (Marton 1979)**

\((R_1, R_2)\) is achievable if

\[
R_1 < I(U_1; Y_1), \\
R_2 < I(U_2; Y_2), \\
R_1 + R_2 < I(U_1; Y_1) + I(U_2; Y_2) - I(U_1; U_2)
\]

for some pmf \(p(u_1, u_2)\) and function \(x(u_1, u_2)\)

- Region is not convex in general; can be convexified via \(Q\)

**Semideterministic BC**

- Marton inner bound tight for **semideterministic BC** \((Y_1 = y_1(X))\): Set \(U_1 = Y_1\)

The capacity region is the set of \((R_1, R_2)\) such that

\[
R_1 \leq H(Y_1), \\
R_2 \leq I(U; Y_2), \\
R_1 + R_2 \leq H(Y_1|U) + I(U; Y_2)
\]

for some \(p(u, x)\)

- **Deterministic BC** \((Y_1 = y_1(X), Y_2 = y_2(X))\): Further set \(U_2 = Y_2\)

The capacity region is the set of \((R_1, R_2)\) such that

\[
R_1 \leq H(Y_1), \\
R_2 \leq H(Y_2), \\
R_1 + R_2 \leq H(Y_1, Y_2)
\]

for some \(p(x)\)
Example: Blackwell channel

Proof of achievability

- Use multicoding and the mutual covering lemma
Proof of achievability

- **Codebook generation**: Fix \( p(u_1, u_2) \) and \( x(u_1, u_2) \)
  
  > For each \( m_1 \in [1 : 2^{nR_1}] \) generate a subcodebook \( C_1(m_1) \) consisting of \( 2^{n(R_1 - R_i)} \) sequences \( u^n_i(l_i) \sim \prod_{i=1}^n p_{U_i}(u_i), l_i \in [(m_i - 1)2^{n(R_1 - R_i)} + 1 : m_i2^{n(R_1 - R_i)}] \)
  
  > Similarly, generate \( C_2(m_2), m_2 \in [1 : 2^{nR_2}] \)

\[
\begin{array}{c|c|c|c|c|c|c}
 & C_1(1) & C_1(2) & \ldots & C_1(m_1) & \ldots & C_1(2^{nR_1}) \\
C_2(1) & u_2^n(1)1 & 2^{n(R_2 - R_1)} & \ldots & \ldots & \ldots & \ldots \\
C_2(2) & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
C_2(2^{nR_2}) & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]
Proof of achievability

- **Encoding:**
  - To send a message pair \((m_1, m_2)\), transmit \(x^n(m_1, m_2)\)

- **Decoding:**
  - Decoder \(j = 1, 2\) finds unique \(\hat{m}_j\) such that \((u^n_j(l_j), y^n_j) \in T^{(\infty)}_e\) for some \(u^n_j(l_j) \in C_j(\hat{m}_j)\)
Analysis of the probability of error

- Consider $P(E)$ conditioned on $(M_1, M_2) = (1, 1)$
- Let $(L_1, L_2)$ denote the pair of chosen indices
- Error events for decoder 1:

  $F_0 = \{ (U^n_1(l_1), U^n_2(l_2)) \notin T^e(n) \text{ for all } (U^n_1(l_1), U^n_2(l_2)) \in C_1(1) \times C_2(1) \}$,
  $E_1 = \{ (U^n_1(L_1), Y^n_1) \notin T^e(n) \}$,
  $E_2 = \{ (U^n_1(l_1), Y^n_1) \in T^e(n) (U_1, Y_1) \text{ for some } l_1 \notin [1 : 2^{n(R_1 - R_1)}] \}$

  Thus, by the union of events bound,
  $P(E_1) \leq P(E_0) + P(E_0^c \cap E_1) + P(E_2)$

Mutual covering lemma ($U_0 = 0$)

- Let $(U_1, U_2) \sim p(u_1, u_2)$ and $\epsilon' < \epsilon$
- For $j = 1, 2$, let $U^n_j(m_j) = \prod_{i=1}^n P_{U_j(u_{ji})}, m_j \in [1 : 2^{nr_j}]$, be pairwise independent
- Assume that $\{U^n_1(m_1)\}$ and $\{U^n_2(m_2)\}$ are independent
Let \((U_1, U_2) \sim p(u_1, u_2)\) and \(\varepsilon' < \varepsilon\).

For \(j = 1, 2\), let \(U^n_j(m_j) \sim \prod_{i=1}^n p_{U_j}(u_{ji}), m_j \in [1 : 2^{nr_j}]\), be pairwise independent.

Assume that \(\{U^n_1(m_1)\}\) and \(\{U^n_2(m_2)\}\) are independent.

### Mutual covering lemma \((U_0 = 0)\)

- Let \((U_1, U_2) \sim p(u_1, u_2)\) and \(\varepsilon' < \varepsilon\).
- For \(j = 1, 2\), let \(U^n_j(m_j) \sim \prod_{i=1}^n p_{U_j}(u_{ji}), m_j \in [1 : 2^{nr_j}]\), be pairwise independent.
- Assume that \(\{U^n_1(m_1)\}\) and \(\{U^n_2(m_2)\}\) are independent.

#### Lemma 8.1 (Mutual covering lemma)

There exists \(\delta(\varepsilon) \to 0\) as \(\varepsilon \to 0\) such that

\[
\lim_{n \to \infty} P\{ (U^n_1(m_1), U^n_2(m_2)) \notin \mathcal{T}_e^{(n)} \text{ for all } m_1 \in [1 : 2^{nr_1}], m_2 \in [1 : 2^{nr_2}] \} = 0
\]

if \(r_1 + r_2 > I(U_1; U_2) + \delta(\varepsilon)\)

- Proof: See NIT Appendix 8A

- This lemma extends the covering lemma:
  - For a single \(U^n\) sequence \((r_1 = 0), r_2 > I(U_1; U_2) + \delta(\varepsilon)\) as in the covering lemma
  - Pairwise independence: linear codes for finite field models

#### Analysis of the probability of error

- Error events for decoder 1:

\[
\mathcal{E}_0 = \{(U^n_1(l_1), U^n_2(l_2)) \notin \mathcal{T}_e^{(n)} \text{ for all } (U^n_1(l_1), U^n_2(l_2)) \in C_1(1) \times C_2(1) \},
\]

\[
\mathcal{E}_{11} = \{(U^n_1(L_1), Y^n_1) \notin \mathcal{T}_e^{(n)} \},
\]

\[
\mathcal{E}_{12} = \{(U^n_1(l_1), Y^n_1) \in \mathcal{T}_e^{(n)}(U_1, Y_1) \text{ for some } l_1 \notin [1 : 2^{n(\tilde{R}_1-R_1)}] \}
\]

- By the mutual covering lemma (with \(r_1 = \tilde{R}_1 - R_1\) and \(r_2 = \tilde{R}_2 - R_2\)),
  \(P(\mathcal{E}_0) \to 0\) if \((\tilde{R}_1 - R_1) + (\tilde{R}_2 - R_2) > I(U_1; U_2) + \delta(\varepsilon')\)

- Since \(\mathcal{E}_0^c = \{(U^n_1(L_1), U^n_2(L_2), X^n) \in \mathcal{T}_e^{(n)} \},\n
  by the conditional typicality lemma, \(P(\mathcal{E}_0^c \cap \mathcal{E}_{11}) \to 0\)

- By the packing lemma, \(P(\mathcal{E}_{12}) \to 0\) if \(\tilde{R}_1 < I(U_1; Y_1) - \delta(\varepsilon)\)

- Similarly, \(P(\mathcal{E}_{12}) \to 0\) if \(\tilde{R}_2 < I(U_2; Y_2) + \delta(\varepsilon)\)

- Using Fourier–Motzkin to eliminate \(\tilde{R}_1\) and \(\tilde{R}_2\) completes the proof
Relationship to Gelfand–Pinsker

- Consider the Marton coding scheme
- Fix $p(u_1, u_2)$ and $x(u_1, u_2)$. This defines a pentagon region
- Consider corner point $(R_1 = I(U_1; Y_1) - I(U_1; U_2), R_2 = I(U_2; Y_2))$
- Marton scheme for communicating $M_1$ is equivalent to G–P for $p(y_1|u_1, u_2)p(u_2)$

The corner point $(R_1 = I(U_1; Y_1), R_2 = I(U_2; Y_2) - I(U_1; U_2))$ achieved similarly

The rest of the pentagon region is achieved by time sharing

Application: Gaussian BC

- Decompose $X$ into the sum of independent $X_1 \sim N(0, \alpha P)$ and $X_2 \sim N(0, \tilde{\alpha} P)$
- Send $M_2$ to $Y_2 = X_2 + X_1 + Z_2$: $R_2 < C(\tilde{\alpha} P/(\alpha P + N_2))$ (treat $X_1$ as noise)
- Send $M_1$ to $Y_1 = X_1 + X_2 + Z_1$: $R_1 < C(\alpha P/N_1)$ (writing on dirty paper)
  - Substitute $U_2 = X_2$ and $U_1 = \beta U_2 + X_1, \beta = \alpha P/(\alpha P + N_1)$ in Marton
- This coding scheme works even when $N_1 > N_2$ (unlike superposition coding)
Gaussian vector broadcast channel

- \( Z_1, Z_2 \sim \mathcal{N}(0, I_r) \)
- Average power constraint: \( \sum_{i=1}^{n} x^T(m_1, m_2, i)x(m_1, m_2, i) \leq nP \)
- Channel is not degraded in general (superposition coding not optimal)
- Marton coding (vector writing on dirty paper) is optimal, however

### Capacity region

- \( \mathcal{R}_1: (R_1, R_2) \) such that
  \[
  R_1 < \frac{1}{2} \log \frac{|G_1 K_1 G_1^T + G_1 K_2 G_1^T + I_r|}{|G_1 K_2 G_1^T + I_r|},
  \]
  \[
  R_2 < \frac{1}{2} \log |G_2 K_2 G_2^T + I_r|
  \]
  for some \( K_1, K_2 \geq 0 \) with \( \text{tr}(K_1 + K_2) \leq P \)
- \( \mathcal{R}_2: (R_1, R_2) \) such that
  \[
  R_1 < \frac{1}{2} \log |G_1 K_1 G_1^T + I_r|,
  \]
  \[
  R_2 < \frac{1}{2} \log \frac{|G_2 K_2 G_2^T + G_2 K_1 G_2^T + I_r|}{|G_2 K_1 G_2^T + I_r|}
  \]
  for some \( K_1, K_2 \geq 0 \) with \( \text{tr}(K_1 + K_2) \leq P \)

**Theorem 9.4 (Weingarten–Steinberg–Shamai 2006)**

\( \mathcal{C} \) is the convex closure of \( \mathcal{R}_1 \cup \mathcal{R}_2 \)
Proof of achievability for $R_2$

- Decompose $X$ into the sum of independent $X_1 \sim N(0, K_1)$ and $X_2 \sim N(0, K_2)$
- Send $M_2$ to $Y_2 = G_2X_2 + G_1X_1 + Z_2$: $R_2 < \frac{1}{2} \log \frac{|G_2K_2G_2^T + G_2K_1G_2^T + I_r|}{|G_2K_1G_2^T + I_r|}$
- Send $M_1$ to $Y_1 = G_1X_1 + G_2X_2 + Z_1$: $R_1 < \frac{1}{2} \log |G_1K_1G_1^T + I_r|$ (vector WDP)
- $R_1$ is achieved similarly

Marton’s inner bound with common message

**Theorem 8.4 (Marton 1979, Liang 2005)**

$(R_0, R_1, R_2)$ is achievable if

\[
R_0 + R_1 < I(U_0, U_1; Y_1), \\
R_0 + R_2 < I(U_0, U_2; Y_2), \\
R_0 + R_1 + R_2 < I(U_0, U_1; Y_1) + I(U_2; Y_2|U_0) - I(U_1; U_2|U_0), \\
R_0 + R_1 + R_2 < I(U_1; Y_1|U_0) + I(U_0, U_2; Y_2) - I(U_1; U_2|U_0), \\
2R_0 + R_1 + R_2 < I(U_0, U_1; Y_1) + I(U_0, U_2; Y_2) - I(U_1; U_2|U_0)
\]

for some $p(u_0, u_1, u_2)$ and function $x(u_0, u_1, u_2)$

- Proof of achievability: Superposition coding + Marton coding
- Tight for all classes of BCs with known capacity regions
- Even for $R_0 = 0$, larger than Marton’s inner bound with $U_0 = \emptyset$ (Theorem 8.3)
Nair–El Gamal outer bound

Theorem 8.6 (Nair–El Gamal 2007)

If \((R_0, R_1, R_2)\) is achievable, then

\[
\begin{align*}
R_0 &\leq \min\{I(U_0; Y_1), I(U_0; Y_2)\}, \\
R_0 + R_1 &\leq I(U_0, U_1; Y_1), \\
R_0 + R_2 &\leq I(U_0, U_2; Y_2), \\
R_0 + R_1 + R_2 &\leq I(U_0, U_1; Y_1) + I(U_2; Y_2|U_0, U_1), \\
R_0 + R_1 + R_2 &\leq I(U_1; Y_1|U_0, U_2) + I(U_0, U_2; Y_2)
\end{align*}
\]

for some \(p(u_1)p(u_2)p(u_0|u_1, u_2)\) and function \(x(u_0, u_1, u_2)\)

- Tight for all BCs with known capacity regions that we discussed so far
- Does not coincide with Marton’s inner bound (Jog–Nair 2010)
- Not tight in general (Geng–Gohari–Nair–Yu 2011)

Summary

- Marton’s inner bound:
  - Multidimensional subcodebook generation
  - Generating correlated codewords for independent messages
- Mutual covering lemma
- Connection between Marton coding and Gelfand–Pinsker coding
- Writing on dirty paper achieves the capacity region of the Gaussian vector BC
- Nair–El Gamal outer bound
References


