Lecture Notes 6
Random Processes

- Definition and Simple Examples
- Important Classes of Random Processes
  - IID
  - Random Walk Process
  - Markov Processes
  - Independent Increment Processes
  - Counting processes and Poisson Process
- Mean and Autocorrelation Function
- Gaussian Random Processes
  - Gauss–Markov Process

Random Process

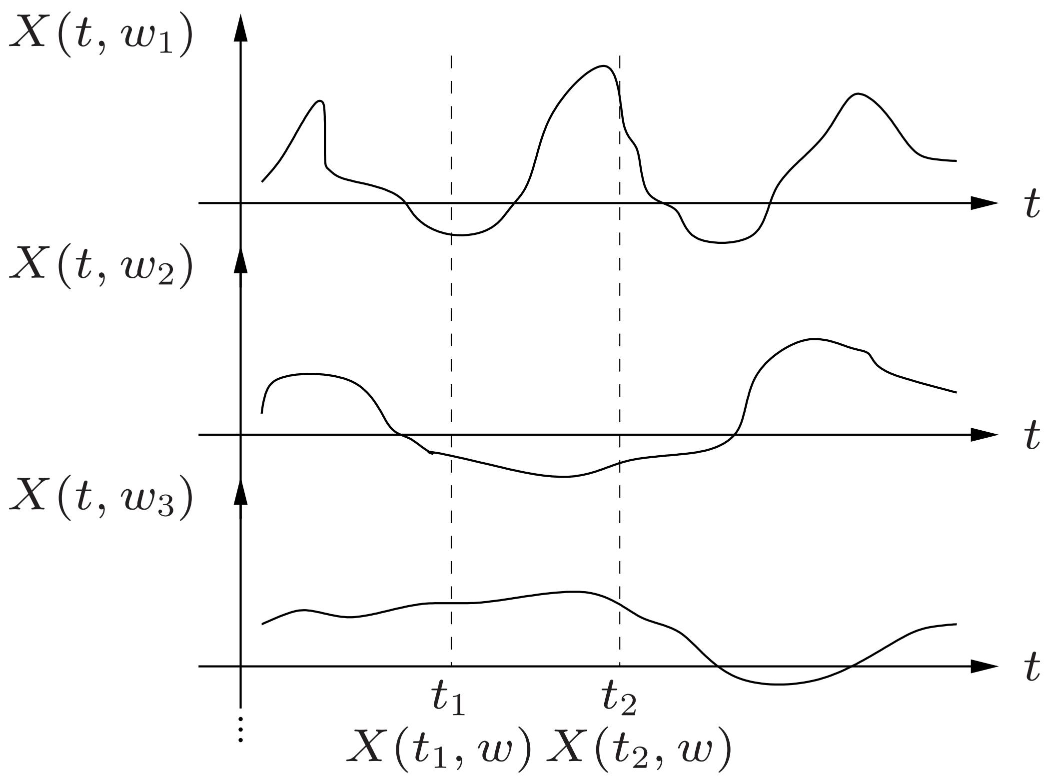
- A random process (RP) (or stochastic process) is an infinite indexed collection of random variables $\{X(t) : t \in \mathcal{T}\}$, defined over a common probability space
- The index parameter t is typically time, but can also be a spatial dimension
- Random processes are used to model random experiments that evolve in time:
  - Received sequence/waveform at the output of a communication channel
  - Packet arrival times at a node in a communication network
  - Thermal noise in a resistor
  - Scores of an NBA team in consecutive games
  - Daily price of a stock
  - Winnings or losses of a gambler

Questions Involving Random Processes

- Dependencies of the random variables of the process
  - How do future received values depend on past received values?
  - How do future prices of a stock depend on its past values?
- Long term averages
  - What is the proportion of time a queue is empty?
  - What is the average noise power at the output of a circuit?
- Extreme or boundary events
  - What is the probability that a link in a communication network is congested?
  - What is the probability that the maximum power in a power distribution line is exceeded?
  - What is the probability that a gambler will lose all his captial?
- Estimation/detection of a signal from a noisy waveform

Two Ways to View a Random Process

- A random process can be viewed as a function $X(t, \omega)$ of two variables, time $t \in \mathcal{T}$ and the outcome of the underlying random experiment $\omega \in \Omega$
  - For fixed t, $X(t, \omega)$ is a random variable over Ω
  - For fixed ω, $X(t, \omega)$ is a deterministic function of t, called a sample function

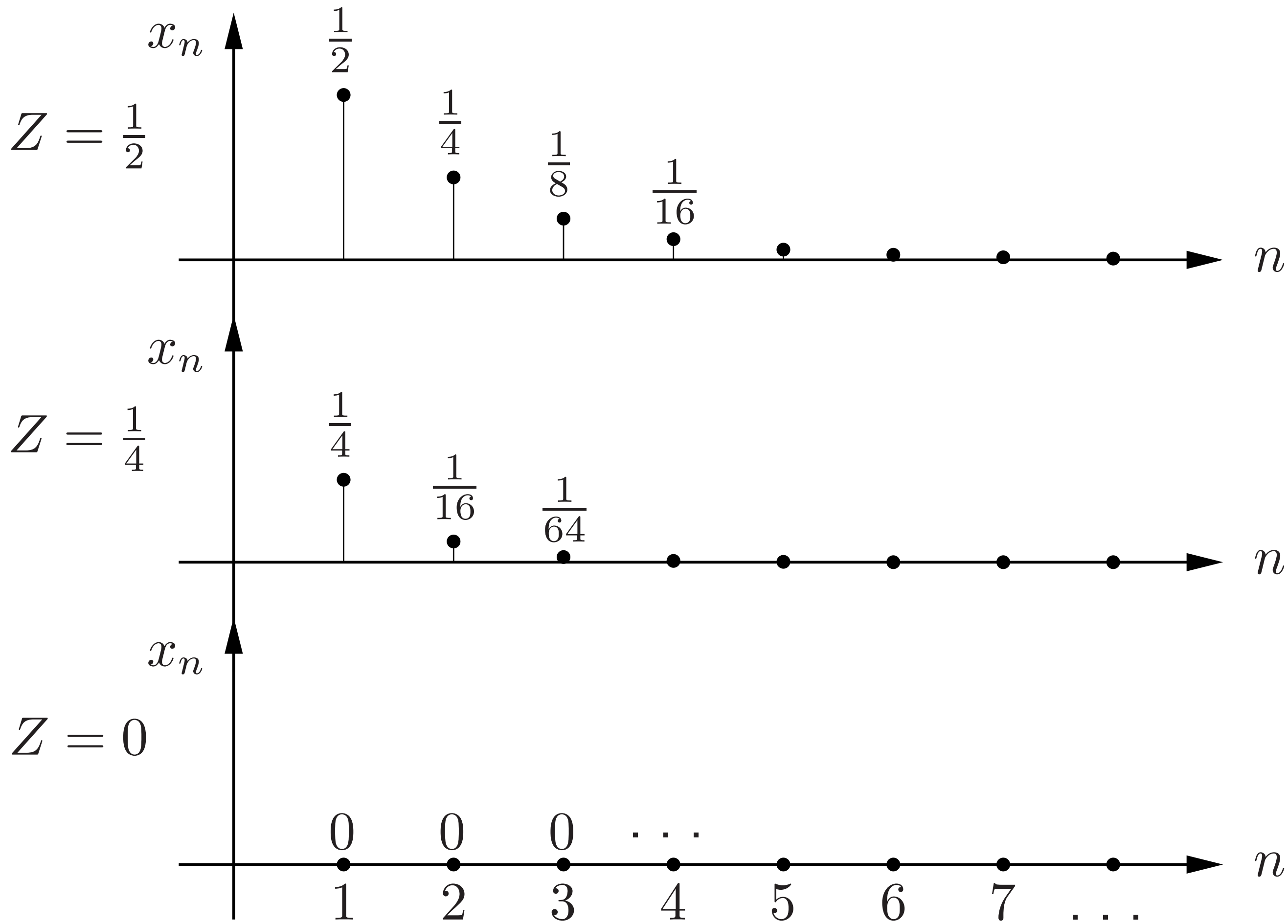


Discrete Time Random Process

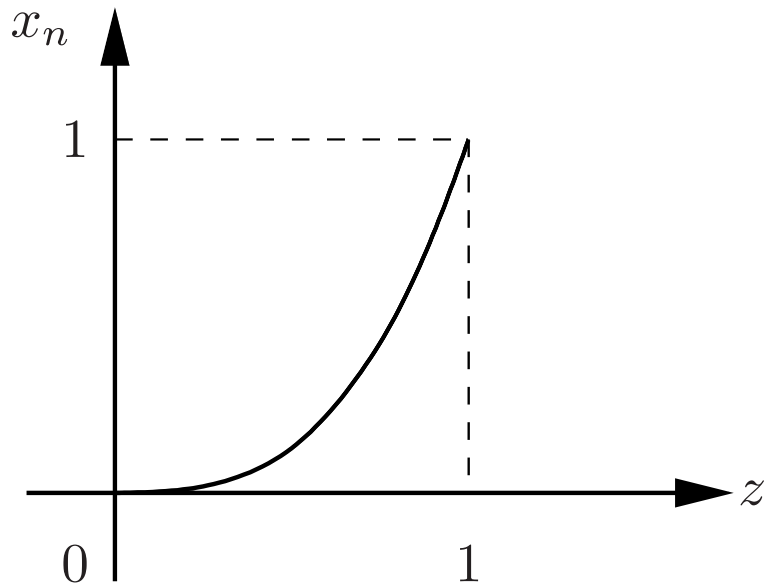
- A random process is said to be discrete time if $\mathcal{T}$ is a countably infinite set, e.g.,
  - $\mathcal{N} = \{0, 1, 2, \ldots\}$
  - $\mathbf{Z} = \{\ldots, -2, -1, 0, +1, +2, \ldots\}$
- In this case the process is denoted by X_n, for $n \in \mathcal{N}$, a countably infinite set, and is simply an infinite sequence of random variables
- A sample function for a discrete time process is called a sample sequence or sample path
- A discrete-time process can comprise discrete, continuous, or mixed r.v.s

Example

- Let $Z \sim \mathrm{U}[0,1]$, and define the discrete time process $X_n = Z^n$ for $n \geq 1$
- Sample paths:



- First-order pdf of the process: For each n, $X_n = Z^n$ is a r.v.; the sequence of pdfs of X_n is called the first-order pdf of the process



Since X_n is a differentiable function of the continuous r.v. Z, we can find its pdf as

$$f_{X_n}(x) = \frac{1}{n x^{(n-1)/n}} = \frac{1}{n} x^{\frac{1}{n}-1}, \quad 0 \le x \le 1$$

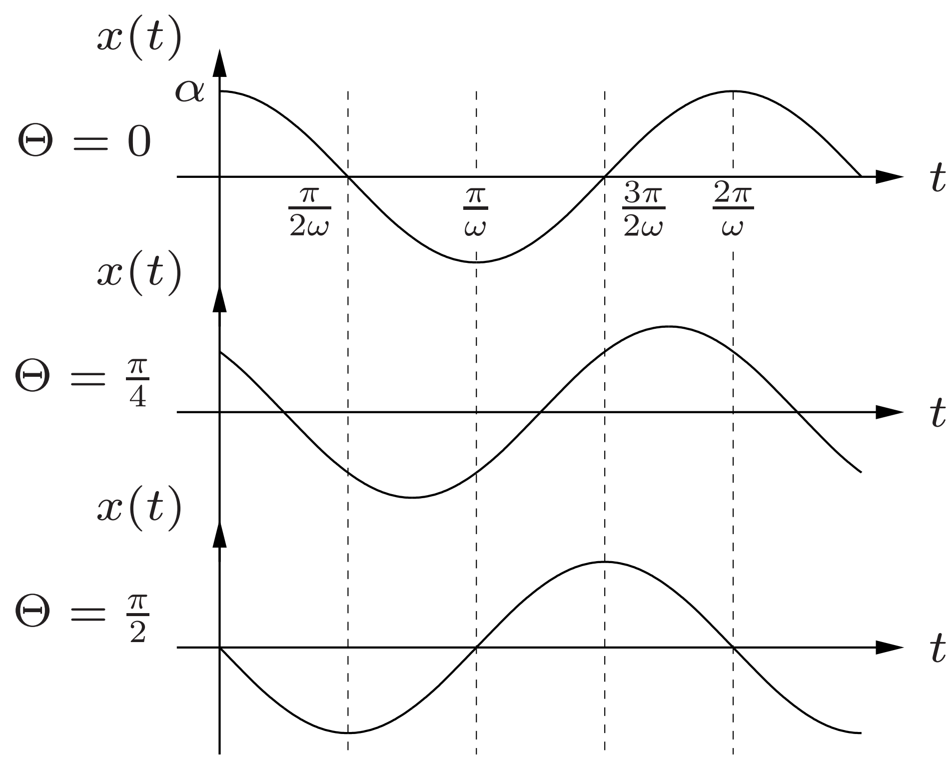
Continuous Time Random Process

- A random process is continuous time if $\mathcal{T}$ is a continuous set
- Example: Sinusoidal Signal with Random Phase

$$X(t) = \alpha \cos(\omega t + \Theta)\,, \quad t \geq 0$$

where $\Theta \sim \mathrm{U}[0, 2\pi]$ and α and ω are constants

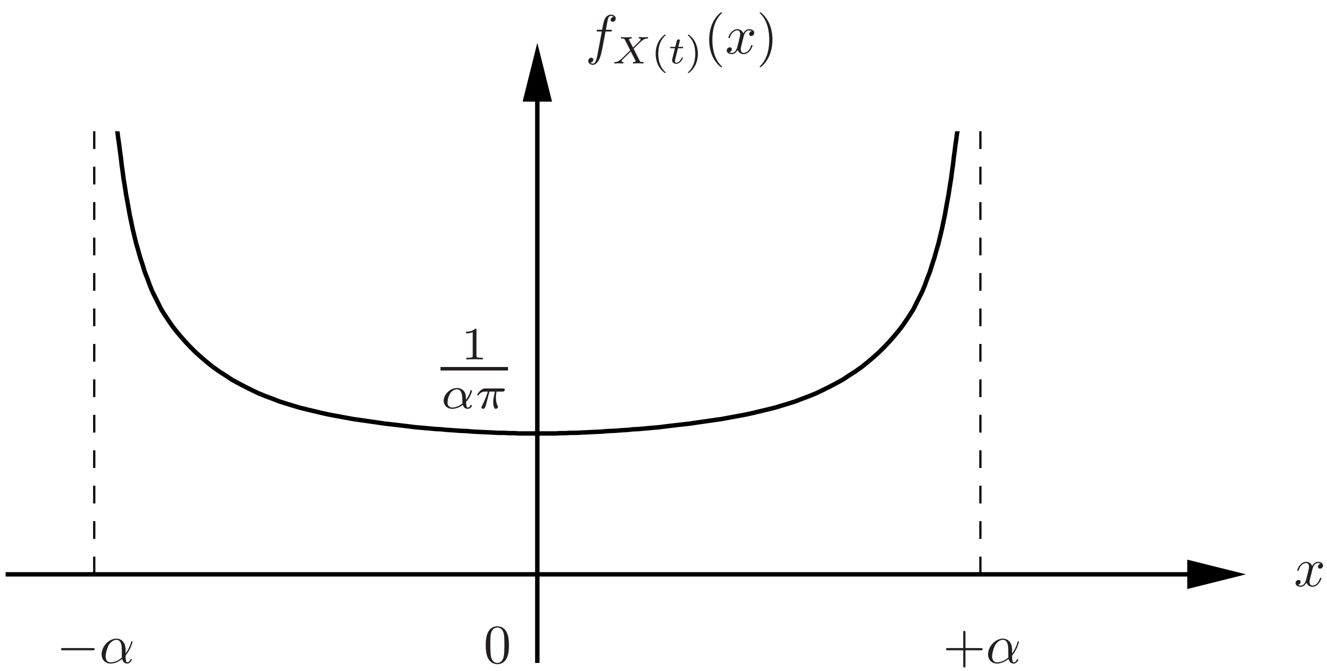
- Sample functions:



- The first-order pdf of the process is the pdf of $X(t) = \alpha\cos(\omega t + \Theta)$. In an earlier homework exercise, we found it to be

$$f_{X(t)}(x) = \frac{1}{\alpha\pi\sqrt{1-(x/\alpha)^2}}, \qquad -\alpha < x < +\alpha$$

The graph of the pdf is shown below



Note that the pdf is independent of t. (The process is stationary)

Specifying a Random Process

- In the above examples we specified the random process by describing the set of sample functions (sequences, paths) and explicitly providing a probability measure over the set of events (subsets of sample functions)

- This way of specifying a random process has very limited applicability, and is suited only for very simple processes

- A random process is typically specified (directly or indirectly) by specifying all its n-th order cdfs (pdfs, pmfs), i.e., the joint cdf (pdf, pmf) of the samples

  $$X(t_1), X(t_2), \ldots, X(t_n)$$

  for every order n and for every set of n points $t_1, t_2, \ldots, t_n \in \mathcal{T}$

- The following examples of important random processes will be specified (directly or indirectly) in this manner

Important Classes of Random Processes

- IID process: $\{X_n : n \in \mathcal{N}\}$ is an IID process if the r.v.s X_n are i.i.d.

  Examples:

  ◦ Bernoulli process: $X_1, X_2, \ldots, X_n, \ldots$ i.i.d. $\sim \mathrm{Bern}(p)$

  ◦ Discrete-time white Gaussian noise (WGN): $X_1, \ldots, X_n, \ldots$ i.i.d. $\sim \mathcal{N}(0, N)$

- Here we specified the n-th order pmfs (pdfs) of the processes by specifying the first-order pmf (pdf) and stating that the r.v.s are independent

- It would be quite difficult to provide the specifications for an IID process by specifying the probability measure over the subsets of the sample space

The Random Walk Process

- Let $Z_1, Z_2, \ldots, Z_n, \ldots$ be i.i.d., where

$$Z_n = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

- The random walk process is defined by

$$X_0 = 0$$

$$X_n = \sum_{i=1}^{n} Z_i\,, \quad n \geq 1$$

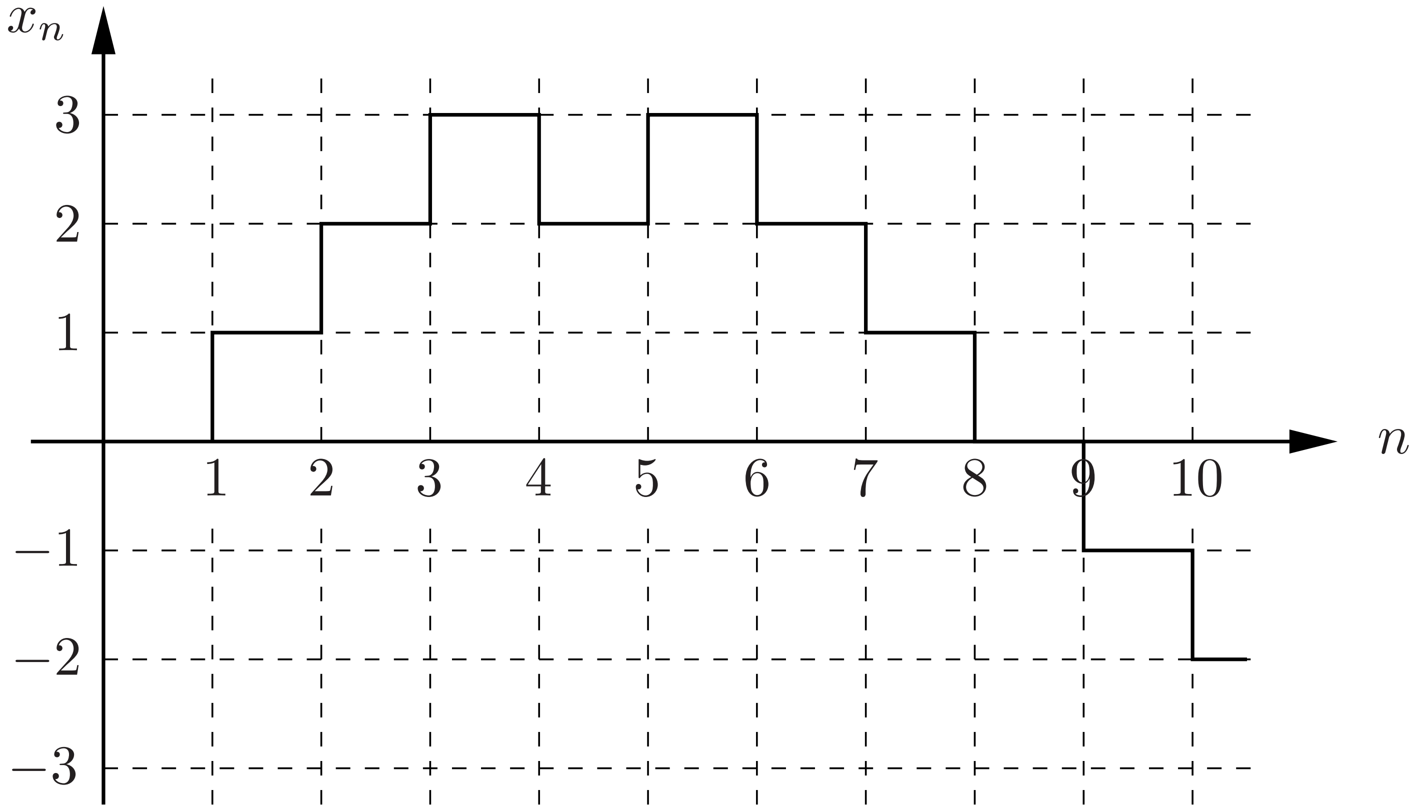
- Again this process is specified by (indirectly) specifying all n-th order pmfs
- Sample path: The sample path for a random walk is a sequence of integers, e.g.,

$$0, +1, 0, -1, -2, -3, -4, \ldots$$

or

$$0, +1, +2, +3, +4, +3, +4, +3, +4, \ldots$$

Example:



z_n : 1 1 1 −1 1 −1 −1 −1 −1 −1

- First-order pmf: The first-order pmf is $\mathrm{P}\{X_n = k\}$ as a function of n. Note that

$$k \in \{-n, -(n-2), \ldots, -2, 0, +2, \ldots, +(n-2), +n\} \quad \text{for } n \text{ even}$$

$$k \in \{-n, -(n-2), \ldots, -1, +1, +3, \ldots, +(n-2), +n\} \quad \text{for } n \text{ odd}$$

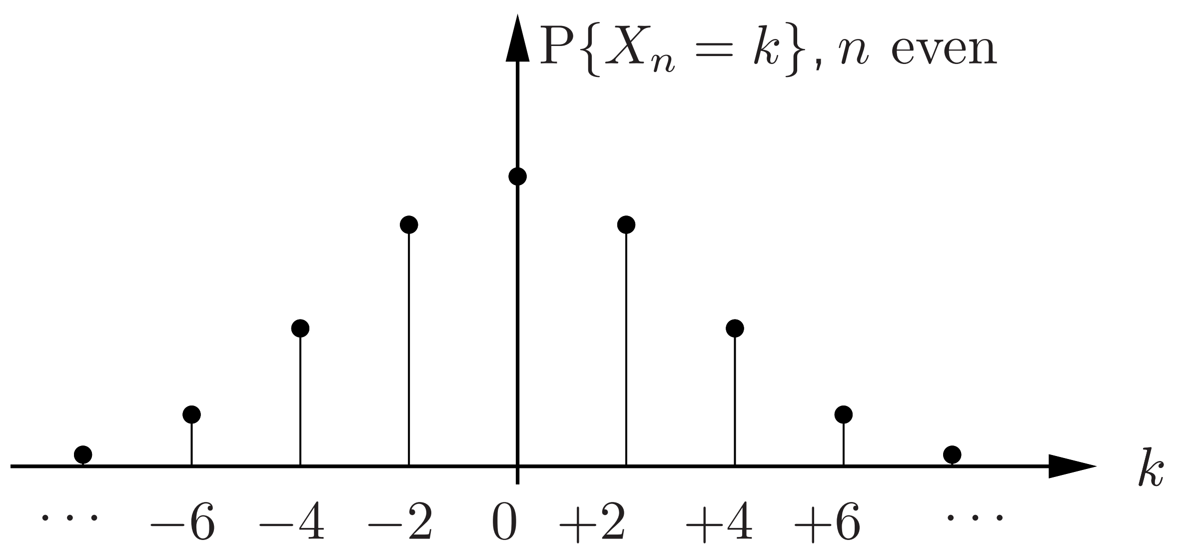
Hence, $\mathrm{P}\{X_n = k\} = 0$ if $n + k$ is odd, or if $k < -n$ or $k > n$

Now for $n+k$ even, let a be the number of $+1$'s in n steps, then the number of -1's is $n-a$, and we find that

$$k = a - (n-a) = 2a - n \;\Rightarrow\; a = \frac{n+k}{2}$$

Thus

$$\begin{aligned} \mathrm{P}\{X_n = k\} &= \mathrm{P}\{\tfrac{1}{2}(n+k) \text{ heads in } n \text{ independent coin tosses}\} \\ &= \binom{n}{\frac{n+k}{2}} \cdot 2^{-n} \quad \text{for } n+k \text{ even and } -n \le k \le n \end{aligned}$$



Markov Processes

- A discrete-time random process X_n is said to be a Markov process if the process future and past are conditionally independent given its present value

- Mathematically this can be rephrased in several ways. For example, if the r.v.s $\{X_n : n \geq 1\}$ are discrete, then the process is Markov iff

$$p_{X_{n+1}|\mathbf{X}^n}(x_{n+1}|x_n, \mathbf{x}^{n-1}) = p_{X_{n+1}|X_n}(x_{n+1}|x_n)$$

  for every n

- IID processes are Markov

- The random walk process is Markov. To see this consider

$$\begin{aligned} \mathrm{P}\{X_{n+1} = x_{n+1} \,|\, \mathbf{X}^n = \mathbf{x}^n\} &= \mathrm{P}\{X_n + Z_{n+1} = x_{n+1} \,|\, \mathbf{X}^n = \mathbf{x}^n\} \\ &= \mathrm{P}\{X_n + Z_{n+1} = x_{n+1} \,|\, X_n = x_n\} \\ &= \mathrm{P}\{X_{n+1} = x_{n+1} \,|\, X_n = x_n\} \end{aligned}$$

Independent Increment Processes

- A discrete-time random process $\{X_n : n \geq 0\}$ is said to be independent increment if the increment random variables

$$X_{n_1},\ X_{n_2} - X_{n_1},\ \ldots,\ X_{n_k} - X_{n_{k-1}}$$

are independent for all sequences of indices such that $n_1 < n_2 < \cdots < n_k$

- Example: Random walk is an independent increment process because

$$X_{n_1} = \sum_{i=1}^{n_1} Z_i\,, \quad X_{n_2} - X_{n_1} = \sum_{i=n_1+1}^{n_2} Z_i\,, \quad \ldots, \quad X_{n_k} - X_{n_{k-1}} = \sum_{i=n_{k-1}+1}^{n_k} Z_i$$

are independent because they are functions of independent random vectors

- The independent increment property makes it easy to find the n-th order pmfs of a random walk process from knowledge only of the first-order pmf

- Example: Find $\mathrm{P}\{X_5 = 3,\, X_{10} = 6,\, X_{20} = 10\}$ for random walk process $\{X_n\}$

  Solution: We use the independent increment property as follows

$$\begin{aligned}
\mathrm{P}\{X_5 = 3,\, X_{10} = 6,\, X_{20} = 10\} &= \mathrm{P}\{X_5 = 3,\, X_{10} - X_5 = 3,\, X_{20} - X_{10} = 4\} \\
&= \mathrm{P}\{X_5 = 3\}\mathrm{P}\{X_5 = 3\}\mathrm{P}\{X_{10} = 4\} \\
&= \binom{5}{4} 2^{-5} \binom{5}{4} 2^{-5} \binom{10}{7} 2^{-10} = 3000 \cdot 2^{-20}
\end{aligned}$$

- In general if a process is independent increment, then it is also Markov. To see this let X_n be an independent increment process and define

$$\mathbf{\Delta X}^n = [X_1,\, X_2 - X_1,\, \ldots,\, X_n - X_{n-1}]^T$$

  Then

$$\begin{aligned}
p_{X_{n+1}|\mathbf{X}^n}(x_{n+1} \,|\, \mathbf{x}^n) &= \mathrm{P}\{X_{n+1} = x_{n+1} \,|\, \mathbf{X}^n = \mathbf{x}^n\} \\
&= \mathrm{P}\{X_{n+1} - X_n + X_n = x_{n+1} \,|\, \mathbf{\Delta X}^n = \mathbf{\Delta x}^n,\, X_n = x_n\} \\
&= \mathrm{P}\{X_{n+1} = x_{n+1} \,|\, X_n = x_n\}
\end{aligned}$$

- The converse is not necessarily true, e.g., IID processes are Markov but not independent increment

- The independent increment property can be extended to continuous-time processes:

  A process $X(t)$, $t \geq 0$, is said to be independent increment if $X(t_1)$, $X(t_2) - X(t_1)$, . . . , $X(t_k) - X(t_{k-1})$ are independent for every $0 \leq t_1 < t_2 < \ldots < t_k$ and every $k \geq 2$

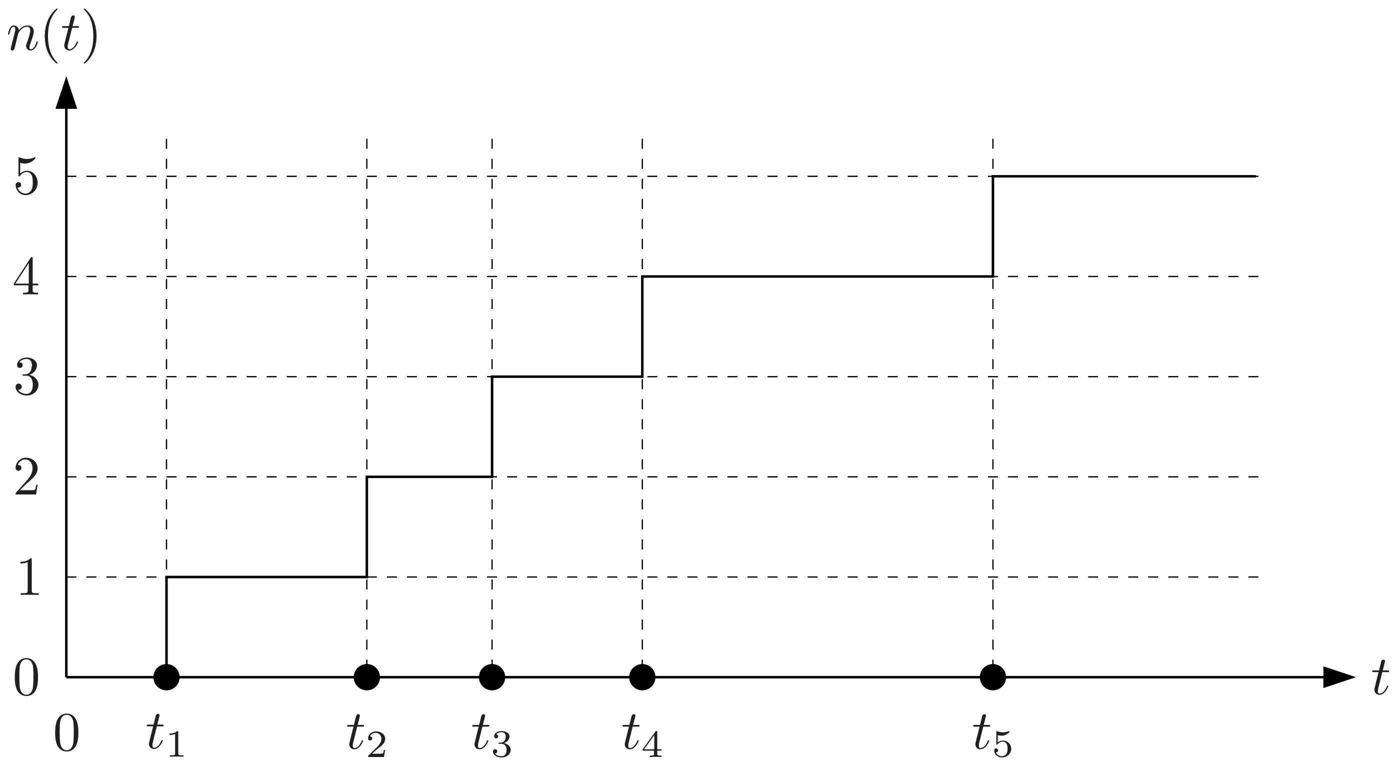
- Markovity can also be extended to continuous-time processes:

  A process $X(t)$ is said to be Markov if $X(t_{k+1})$ and $(X(t_1), \ldots, X(t_{k-1}))$ are conditionally independent given $X(t_k)$ for every $0 \leq t_1 < t_2 < \ldots < t_k < t_{k+1}$ and every $k \geq 2$

Counting Processes and Poisson Process

- A continuous-time random process $N(t)$, $t \geq 0$, is said to be a counting process if $N(0) = 0$ and $N(t) = n$, $n \in \{0, 1, 2, \ldots\}$, is the number of events from 0 to t (hence $N(t_2) \geq N(t_1)$ for every $t_2 > t_1 \geq 0$)

  Sample path of a counting process:



  $t_1, t_2, \ldots$ are the arrival times or the wait times of the events

  $t_1, t_2 - t_1, \ldots$ are the interarrival times of the events

The events may be:

- Photon arrivals at an optical detector
- Packet arrivals at a router
- Student arrivals at a class

- The Poisson process is a counting process in which the events are "independent of each other"

- More precisely, $N(t)$ is a Poisson process with rate (intensity) $\lambda > 0$ if:
  - $N(0) = 0$
  - $N(t)$ is independent increment
  - $(N(t_2) - N(t_1)) \sim \mathrm{Poisson}(\lambda(t_2 - t_1))$ for all $t_2 > t_1 \geq 0$

- To find the kth order pmf, we use the independent increment property

$$\begin{aligned}
&\mathrm{P}\{N(t_1) = n_1, N(t_2) = n_2, \ldots, N(t_k) = n_k\} \\
&\quad = \mathrm{P}\{N(t_1) = n_1, N(t_2) - N(t_1) = n_2 - n_1, \ldots, N(t_k) - N(t_{k-1}) = n_k - n_{k-1}\} \\
&\quad = p_{N(t_1)}(n_1) p_{N(t_2)-N(t_1)}(n_2 - n_1) \ldots p_{N(t_k)-N(t_{k-1})}(n_k - n_{k-1})
\end{aligned}$$

- Example: Packets arrive at a router according to a Poisson process $N(t)$ with rate λ. Assume the service time for each packet $T \sim \mathrm{Exp}(\beta)$ is independent of $N(t)$ and of each other. What is the probability that k packets arrive during a service time?

- Merging: The sum of independent Poisson process is Poisson. This is a consequence of the infinite divisibility of the Poisson r.v.

- Branching: Let $N(t)$ be a Poisson process with rate λ. We split $N(t)$ into two counting subprocesses $N_1(t)$ and $N_2(t)$ such that $N(t) = N_1(t) + N_2(t)$ as follows:

  Each event is randomly and independently assigned to process $N_1(t)$ with probability p, otherwise it is assigned to $N_2(t)$

  Then $N_1(t)$ is a Poisson process with rate $p\lambda$ and $N_2(t)$ is a Poisson process with rate $(1-p)\lambda$

  This can be generalized to splitting a Poisson process into more than two processes

Related Processes

- Arrival time process: Let $N(t)$ be Poisson with rate λ. The arrival time process T_n, $n \geq 0$ is a discrete time process such that:
  - $T_0 = 0$
  - T_n is the arrival time of the nth event of $N(t)$
- Interarrival time process: Let $N(t)$ be a Poisson process with rate λ. The interarrival time process is $X_n = T_n - T_{n-1}$ for $n \geq 1$
- X_n is an IID process with $X_n \sim \mathrm{Exp}(\lambda)$
- $T_n = \sum_{i=1}^{n} X_i$ is an independent increment process with $T_{n_2} - T_{n_1} \sim$ Gamma$(\lambda, n_2 - n_1)$ for $n_2 > n_1 \geq 1$, i.e.,

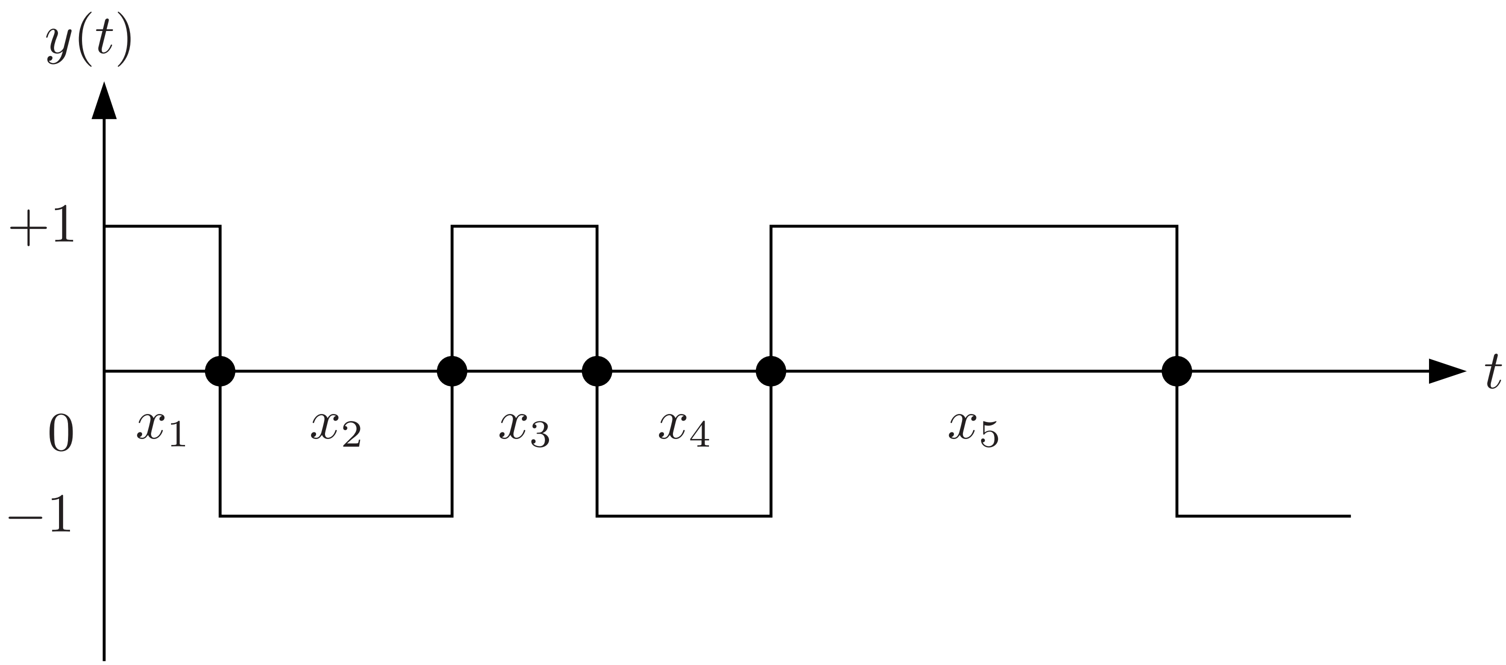
$$f_{T_{n_2}-T_{n_1}}(t) = \frac{\lambda^{n_2-n_1} t^{n_2-n_1-1}}{(n_2-n_1-1)!} e^{-\lambda t}$$

- Example: Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2, respectively. What is the probability that $N_1(t) = 1$ before $N_2(t) = 1$?

- Random telegraph process: A random telegraph process $Y(t)$, $t \geq 0$, assumes values of $+1$ and -1 with $Y(0) = +1$ with probability $1/2$ and -1 with probability $1/2$, and

  $Y(t)$ changes polarities with each event of a Poisson process with rate $\lambda > 0$

  Sample path:



Mean and Autocorrelation Functions

- For a random vector $\mathbf{X}$ the first and second order moments are
  - mean $\boldsymbol{\mu} = \mathrm{E}(\mathbf{X})$
  - correlation matrix $R_{\mathbf{X}} = \mathrm{E}(\mathbf{X}\mathbf{X}^T)$
- For a random process $X(t)$ the first and second order moments are
  - mean function: $\mu_X(t) = \mathrm{E}(X(t))$ for $t \in \mathcal{T}$
  - autocorrelation function: $R_X(t_1, t_2) = \mathrm{E}\left(X(t_1)X(t_2)\right)$ for $t_1, t_2 \in \mathcal{T}$
- The autocovariance function of a random process is defined as

$$C_X(t_1, t_2) = \mathrm{E}\left[\left(X(t_1) - \mathrm{E}(X(t_1))\right)\left(X(t_2) - \mathrm{E}(X(t_2))\right)\right]$$

The autocovariance function can be expressed using the mean and autocorrelation functions as

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Examples

- IID process:

$$\mu_X(n) = \mathrm{E}(X_1)$$
$$R_X(n_1, n_2) = \mathrm{E}(X_{n_1} X_{n_2}) \;=\; \begin{cases} \mathrm{E}(X_1^2) & n_1 = n_2 \\ (\mathrm{E}(X_1))^2 & n_1 \neq n_2 \end{cases}$$

- Random phase signal process:

$$\mu_X(t) = \mathrm{E}(\alpha \cos(\omega t + \Theta)) \;=\; \int_0^{2\pi} \frac{\alpha}{2\pi} \cos(\omega t + \theta)\, d\theta \;=\; 0$$

$$\begin{aligned} R_X(t_1, t_2) &= \mathrm{E}\left(X(t_1) X(t_2)\right) \\ &= \int_0^{2\pi} \frac{\alpha^2}{2\pi} \cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta)\, d\theta \\ &= \int_0^{2\pi} \frac{\alpha^2}{4\pi} \big[\cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2)) \big]\, d\theta \\ &= \frac{\alpha^2}{2} \cos(\omega(t_1 - t_2)) \end{aligned}$$

- Random walk:

$$\mu_X(n) = \mathrm{E}\left(\sum_{i=1}^{n} Z_i\right) = \sum_{i=1}^{n} 0 = 0$$

$$\begin{aligned}
R_X(n_1, n_2) &= \mathrm{E}(X_{n_1} X_{n_2}) \\
&= \mathrm{E}\left[X_{n_1}(X_{n_2} - X_{n_1} + X_{n_1})\right] \\
&= \mathrm{E}(X_{n_1}^2) = n_1 \quad \text{assuming } n_2 \ge n_1 \\
&= \min\{n_1, n_2\} \quad \text{in general}
\end{aligned}$$

- Poisson process:

$$\begin{aligned}
\mu_N(t) &= \lambda t \\
R_N(t_1, t_2) &= \mathrm{E}(N(t_1)N(t_2)) \\
&= \mathrm{E}\left[N(t_1)(N(t_2) - N(t_1) + N(t_1))\right] \\
&= \lambda t_1 \times \lambda(t_2 - t_1) + \lambda t_1 + \lambda^2 t_1^2 = \lambda t_1 + \lambda^2 t_1 t_2 \quad \text{assuming } t_2 \ge t_1 \\
&= \lambda \min\{t_1, t_2\} + \lambda^2 t_1 t_2
\end{aligned}$$

Gaussian Random Processes

- A Gaussian random process (GRP) is a random process $X(t)$ such that

$$[X(t_1),\, X(t_2),\, \ldots,\, X(t_n)\,]^T$$

  is a GRV for all $t_1, t_2, \ldots, t_n \in \mathcal{T}$

- Since the joint pdf for a GRV is specified by its mean and covariance matrix, a GRP is specified by its mean $\mu_X(t)$ and autocorrelation $R_X(t_1, t_2)$ functions

- Example: The discrete time WGN process is a GRP

Gauss-Markov Process

- Let Z_n, $n \geq 1$, be a WGN process, i.e., an IID process with $Z_1 \sim \mathcal{N}(0, N)$

  The Gauss-Markov process is a first-order autoregressive process defined by

$$X_1 = Z_1$$
$$X_n = \alpha X_{n-1} + Z_n\,, \quad n > 1\,,$$

  where $|\alpha| < 1$

- This process is a GRP, since $X_1 = Z_1$ and $X_k = \alpha X_{k-1} + Z_k$ where $Z_1, Z_2, \ldots$ are i.i.d. $\mathcal{N}(0, N)$,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{n-2} & \alpha^{n-3} & \cdots & 1 & 0 \\ \alpha^{n-1} & \alpha^{n-2} & \cdots & \alpha & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix}$$

  is a linear transformation of a GRV and is therefore a GRV

- Clearly, the Gauss-Markov process is Markov. It is not, however, an independent increment process

- Mean and covariance functions:

$$\begin{aligned}\mu_X(n) &= \mathrm{E}(X_n) \;=\; \mathrm{E}(\alpha X_{n-1} + Z_n) \\ &= \alpha\,\mathrm{E}(X_{n-1}) + \mathrm{E}(Z_n) \;=\; \alpha\,\mathrm{E}(X_{n-1}) = \alpha^{n-1}\,\mathrm{E}(Z_1) \;=\; 0\end{aligned}$$

To find the autocorrelation function, for $n_2 > n_1$ we write

$$X_{n_2} = \alpha^{n_2-n_1} X_{n_1} + \sum_{i=0}^{n_2-n_1-1} \alpha^i Z_{n_2-i}$$

Thus

$$R_X(n_1,n_2) \;=\; \mathrm{E}(X_{n_1}X_{n_2}) \;=\; \alpha^{n_2-n_1}\,\mathrm{E}(X_{n_1}^2) + 0\,,$$

since X_{n_1} and Z_{n_2-i} are independent, zero mean for $0 \le i \le n_2 - n_1 - 1$

Next, to find $\mathrm{E}(X_{n_1}^2)$, consider

$$\begin{aligned}\mathrm{E}(X_1^2) &= N \\ \mathrm{E}(X_{n_1}^2) &= \mathrm{E}\left[(\alpha X_{n_1-1} + Z_{n_1})^2\right] \;=\; \alpha^2\,\mathrm{E}(X_{n_1-1}^2) + N\end{aligned}$$

Thus

$$\mathrm{E}(X_{n_1}^2) = \frac{1-\alpha^{2n_1}}{1-\alpha^2} N$$

Finally the autocorrelation function is

$$R_X(n_1, n_2) = \alpha^{|n_2 - n_1|} \frac{1 - \alpha^{2\min\{n_1, n_2\}}}{1 - \alpha^2} N$$

- Estimation of Gauss-Markov process: Suppose we observe a noisy version of the Gauss-Markov process,

  $$Y_n = X_n + W_n,$$

  where W_n is a WGN process independent of Z_n with average power Q

  We can use the Kalman filter from Lecture Notes 4 to estimate X_{i+1} from Y^i as follows:

  Initialization:

  $$\hat{X}_{1|0} = 0$$
  $$\sigma^2_{1|0} = N$$

Update: For $i = 2, 3, \ldots,$

$$\hat{X}_{i+1|i} = \alpha \hat{X}_{i|i-1} + k_i(Y_i - \hat{X}_{i|i-1}),$$

$$\sigma^2_{i+1|i} = \alpha(\alpha - k_i)\sigma^2_{i|i-1} + N, \text{ where}$$

$$k_i = \frac{\alpha \sigma^2_{i|i-1}}{\sigma^2_{i|i-1} + Q}$$

Substituting from the k_i equation into the MSE update equation, we obtain

$$\sigma^2_{i+1|i} = \frac{\alpha^2 Q \sigma^2_{i|i-1}}{\sigma^2_{i|i-1} + Q} + N,$$

This is a Riccati recursion (a quadratic recursion in the MSE) and has a steady state solution:

$$\sigma^2 = \frac{\alpha^2 Q \sigma^2}{\sigma^2 + Q} + N$$

Solving this quadratic equation, we obtain

$$\sigma^2 = \frac{N - (1 - \alpha^2)Q + \sqrt{4NQ + (N - (1 - \alpha^2)Q)^2}}{2}$$

The Kalman gain k_i converges to

$$k = \frac{-N - (1-\alpha^2)Q + \sqrt{4NQ + (N - (1-\alpha^2)Q)^2}}{2\alpha Q}$$

and the steady-state Kalman filter is

$$\hat{X}_{i+1|i} = \alpha \hat{X}_{i|i-1} + k(Y_i - \hat{X}_{i|i-1})$$