Lecture Notes 1
Probability and Random Variables

- Probability Spaces
- Conditional Probability and Independence
- Random Variables
- Functions of a Random Variable
- Generation of a Random Variable
- Jointly Distributed Random Variables
- Scalar detection

Probability Theory

- Probability theory provides the mathematical rules for assigning probabilities to outcomes of random experiments, e.g., coin flips, packet arrivals, noise voltage
- Basic elements of probability theory:
  - Sample space Ω: set of all possible "elementary" or "finest grain" outcomes of the random experiment
  - Set of events $\mathcal{F}$: set of (all?) subsets of Ω — an event $A \subset \Omega$ occurs if the outcome $\omega \in A$
  - Probability measure P: function over $\mathcal{F}$ that assigns probabilities to events according to the axioms of probability (see below)
- Formally, a probability space is the triple $(\Omega, \mathcal{F}, \mathrm{P})$

Axioms of Probability

- A probability measure P satisfies the following axioms:
  1. $\mathrm{P}(A) \geq 0$ for every event A in $\mathcal{F}$
  2. $\mathrm{P}(\Omega) = 1$
  3. If $A_1, A_2, \ldots$ are disjoint events—i.e., $A_i \cap A_j = \emptyset$, for all $i \neq j$—then

$$\mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{P}(A_i)$$

- Notes:
  - P is a measure in the same sense as mass, length, area, and volume—all satisfy axioms 1 and 3
  - Unlike these other measures, P is bounded by 1 (axiom 2)
  - This analogy provides some intuition but is not sufficient to fully understand probability theory—other aspects such as conditioning and independence are unique to probability

Discrete Probability Spaces

- A sample space Ω is said to be discrete if it is countable
- Examples:
  - Rolling a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - Flipping a coin n times: $\Omega = \{H, T\}^n$, sequences of heads/tails of length n
  - Flipping a coin until the first heads occurs: $\Omega = \{H, TH, TTH, TTTH, \ldots\}$
- For discrete sample spaces, the set of events $\mathcal{F}$ can be taken to be the set of all subsets of Ω, sometimes called the power set of Ω
- Example: For the coin flipping experiment,

$$\mathcal{F} = \{\emptyset,\ \{H\},\ \{T\},\ \Omega\}$$

- $\mathcal{F}$ does not have to be the entire power set (more on this later)

- The probability measure P can be defined by assigning probabilities to individual outcomes — single outcome events $\{\omega\}$ — so that:

$$\mathrm{P}(\{\omega\}) \geq 0 \text{ for every } \omega \in \Omega$$

$$\sum_{\omega \in \Omega} \mathrm{P}(\{\omega\}) = 1$$

- The probability of any other event A is simply

$$\mathrm{P}(A) = \sum_{\omega \in A} \mathrm{P}(\{\omega\})$$

- Example: For the die rolling experiment, assign

$$\mathrm{P}(\{i\}) = \tfrac{1}{6} \quad \text{for } i = 1, 2, \ldots, 6$$

The probability of the event "the outcome is even," $A = \{2, 4, 6\}$, is

$$\mathrm{P}(A) = \mathrm{P}(\{2\}) + \mathrm{P}(\{4\}) + \mathrm{P}(\{6\}) = \tfrac{3}{6} = \tfrac{1}{2}$$

Continuous Probability Spaces

- A continuous sample space Ω has an uncountable number of elements
- Examples:
  - Random number between 0 and 1: $\Omega = (0, 1\,]$
  - Point in the unit disk: $\Omega = \{(x, y) \colon x^2 + y^2 \leq 1\}$
  - Arrival times of n packets: $\Omega = (0, \infty)^n$
- For continuous Ω, we cannot in general define the probability measure P by first assigning probabilities to outcomes
- To see why, consider assigning a uniform probability measure over $(0, 1\,]$
  - In this case the probability of each single outcome event is zero
  - How do we find the probability of an event such as $A = [0.25, 0.75]$?

- Another difference for continuous Ω: we cannot take the set of events $\mathcal{F}$ as the power set of Ω. (To learn why you need to study measure theory, which is beyond the scope of this course)

- The set of events $\mathcal{F}$ cannot be an arbitrary collection of subsets of Ω. It must make sense, e.g., if A is an event, then its complement A^c must also be an event, the union of two events must be an event, and so on

- Formally, $\mathcal{F}$ must be a sigma algebra (σ-algebra, σ-field), which satisfies the following axioms:

  1. $\emptyset \in \mathcal{F}$
  2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
  3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

- Of course, the power set is a sigma algebra. But we can define smaller σ-algebras. For example, for rolling a die, we could define the set of events as

$$\mathcal{F} = \{\emptyset, \text{odd}, \text{even}, \Omega\}$$

- For $\Omega = R = (-\infty, \infty)$ (or $(0, \infty)$, $(0, 1)$, etc.) $\mathcal{F}$ is typically defined as the family of sets obtained by starting from the intervals and taking countable unions, intersections, and complements

- The resulting $\mathcal{F}$ is called the Borel field

- Note: Amazingly there are subsets in R that cannot be generated in this way! (Not ones that you are likely to encounter in your life as an engineer or even as a mathematician)

- To define a probability measure over a Borel field, we first assign probabilities to the intervals in a consistent way, i.e., in a way that satisfies the axioms of probability

  For example to define uniform probability measure over $(0, 1)$, we first assign $\mathrm{P}((a, b)) = b - a$ to all intervals

- In EE 278 we do not deal with sigma fields or the Borel field beyond (kind of) knowing what they are

Useful Probability Laws

- Union of Events Bound:

$$\mathrm{P}\Big(\bigcup_{i=1}^{n} A_i\Big) \le \sum_{i=1}^{n} \mathrm{P}(A_i)$$

- Law of Total Probability: Let $A_1, A_2, A_3, \ldots$ be events that partition Ω, i.e., disjoint ($A_i \cap A_j = \emptyset$ for $i \neq j$) and $\bigcup_i A_i = \Omega$. Then for any event B

$$\mathrm{P}(B) = \textstyle\sum_i \mathrm{P}(A_i \cap B)$$

The Law of Total Probability is very useful for finding probabilities of sets

Conditional Probability

- Let B be an event such that $\mathrm{P}(B) \neq 0$. The conditional probability of event A given B is defined to be

$$\mathrm{P}(A \,|\, B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A, B)}{\mathrm{P}(B)}$$

- The function $\mathrm{P}(\cdot \,|\, B)$ is a probability measure over $\mathcal{F}$, i.e., it satisfies the axioms of probability

- Chain rule: $\mathrm{P}(A, B) = \mathrm{P}(A)\mathrm{P}(B \,|\, A) = \mathrm{P}(B)\mathrm{P}(A \,|\, B)$ (this can be generalized to n events)

- The probability of event A given B, a nonzero probability event — the a posteriori probability of A — is related to the unconditional probability of A — the a priori probability — by

$$\mathrm{P}(A \,|\, B) = \frac{\mathrm{P}(B \,|\, A)}{\mathrm{P}(B)} \mathrm{P}(A)$$

  This follows directly from the definition of conditional probability

Bayes Rule

- Let $A_1, A_2, \ldots, A_n$ be nonzero probability events that partition Ω, and let B be a nonzero probability event
- We know $\mathrm{P}(A_i)$ and $\mathrm{P}(B \mid A_i)$, $i = 1, 2, \ldots, n$, and want to find the a posteriori probabilities $\mathrm{P}(A_j \mid B)$, $j = 1, 2, \ldots, n$
- We know that
$$\mathrm{P}(A_j \mid B) = \frac{\mathrm{P}(B \mid A_j)}{\mathrm{P}(B)} \mathrm{P}(A_j)$$
- By the law of total probability
$$\mathrm{P}(B) = \sum_{i=1}^{n} \mathrm{P}(A_i, B) = \sum_{i=1}^{n} \mathrm{P}(A_i)\mathrm{P}(B \mid A_i)$$
- Substituting, we obtain Bayes rule
$$\mathrm{P}(A_j \mid B) = \frac{\mathrm{P}(B \mid A_j)}{\sum_{i=1}^{n} \mathrm{P}(A_i)\mathrm{P}(B \mid A_i)} \mathrm{P}(A_j), \; j = 1, 2, \ldots, n$$
- Bayes rule also applies to a (countably) infinite number of events

Independence

- Two events are said to be statistically independent if

$$\mathrm{P}(A, B) = \mathrm{P}(A)\mathrm{P}(B)$$

- When $\mathrm{P}(B) \neq 0$, this is equivalent to

$$\mathrm{P}(A \,|\, B) = \mathrm{P}(A)$$

  In other words, knowing whether B occurs does not change the probability of A

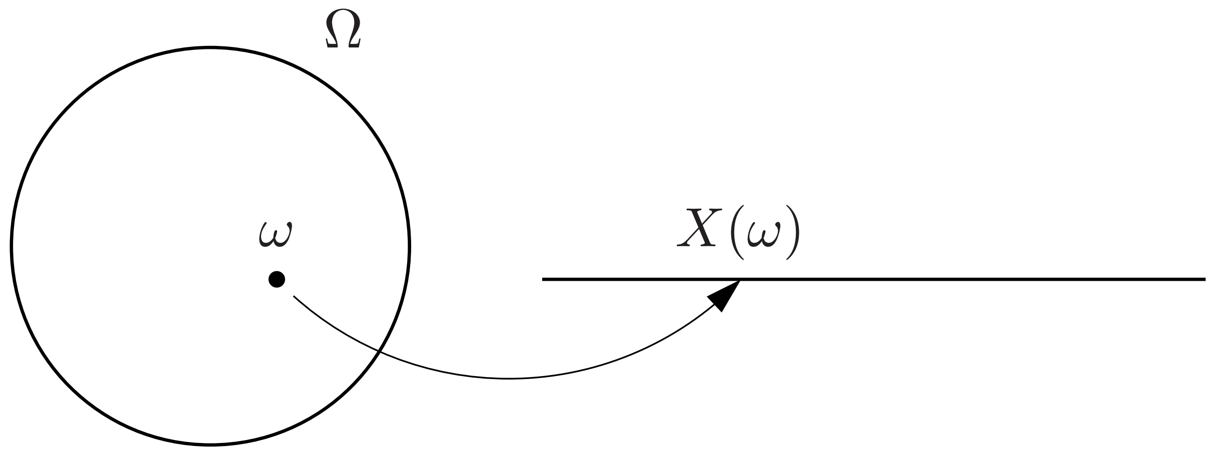
- The events $A_1, A_2, \ldots, A_n$ are said to be independent if for every subset $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ of the events,

$$\mathrm{P}(A_{i_1}, A_{i_2}, \ldots, A_{i_k}) \; = \prod_{j=1}^{k} \mathrm{P}(A_{i_j})$$

- Note: $\mathrm{P}(A_1, A_2, \ldots, A_n) = \prod_{j=1}^{n} \mathrm{P}(A_i)$ is not sufficient for independence

Random Variables

- A random variable (r.v.) is a real-valued function $X(\omega)$ over a sample space Ω, i.e., $X : \Omega \to \mathbf{R}$

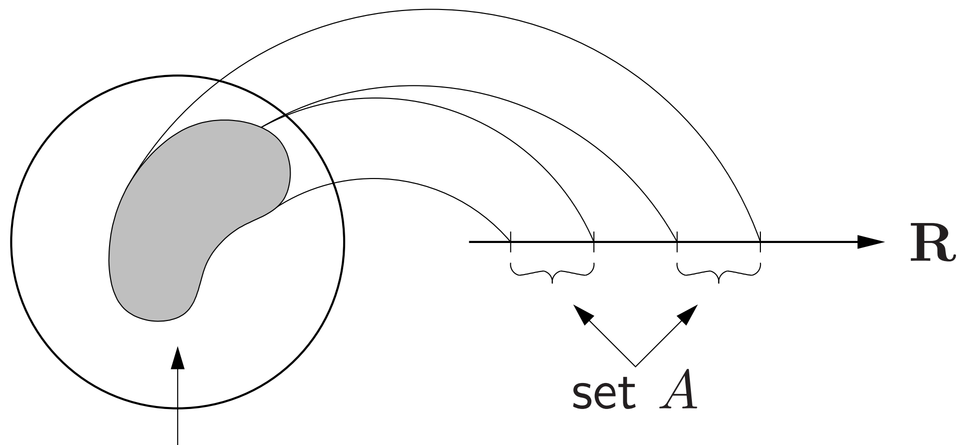


- Notations:
  - We use upper case letters for random variables: X, Y, Z, Φ, Θ, . . .
  - We use lower case letters for values of random variables: $X = x$ means that random variable X takes on the value x, i.e., $X(\omega) = x$ where ω is the outcome

Specifying a Random Variable

- Specifying a random variable means being able to determine the probability that $X \in A$ for any Borel set $A \subset \mathbf{R}$, in particular, for any interval $(a, b]$

- To do so, consider the inverse image of A under X, i.e., $\{\omega : X(\omega) \in A\}$



inverse image of A under $X(\omega)$, i.e., $\{\omega : X(\omega) \in A\}$

- Since $X \in A$ iff $\omega \in \{\omega : X(\omega) \in A\}$,

$$\mathrm{P}(\{X \in A\}) = \mathrm{P}(\{\omega : X(\omega) \in A\}) = \mathrm{P}\{\omega : X(\omega) \in A\}$$

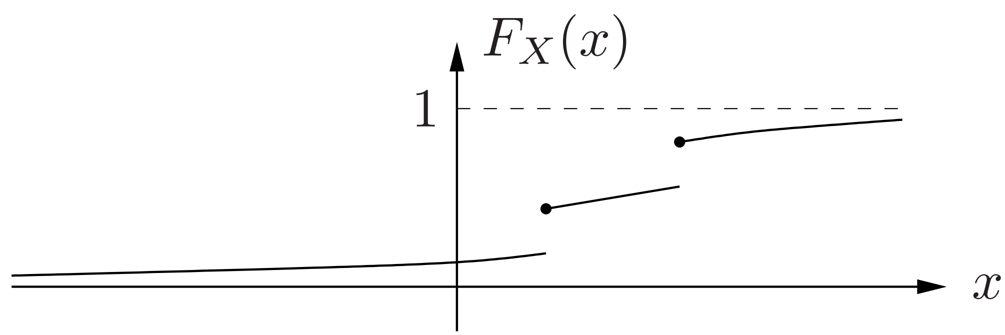
Shorthand: $\mathrm{P}(\{\text{set description}\}) = \mathrm{P}\{\text{set description}\}$

Cumulative Distribution Function (CDF)

- We need to be able to determine $\mathrm{P}\{X \in A\}$ for any Borel set $A \subset \mathbf{R}$, i.e., any set generated by starting from intervals and taking countable unions, intersections, and complements
- Hence, it suffices to specify $\mathrm{P}\{X \in (a, b]\}$ for all intervals. The probability of any other Borel set can be determined by the axioms of probability
- Equivalently, it suffices to specify its cumulative distribution function (cdf):

$$F_X(x) = \mathrm{P}\{X \leq x\} = \mathrm{P}\{X \in (-\infty, x]\}, \quad x \in \mathbf{R}$$

- Properties of cdf:
  - $F_X(x) \geq 0$
  - $F_X(x)$ is monotonically nondecreasing, i.e., if $a > b$ then $F_X(a) \geq F_X(b)$

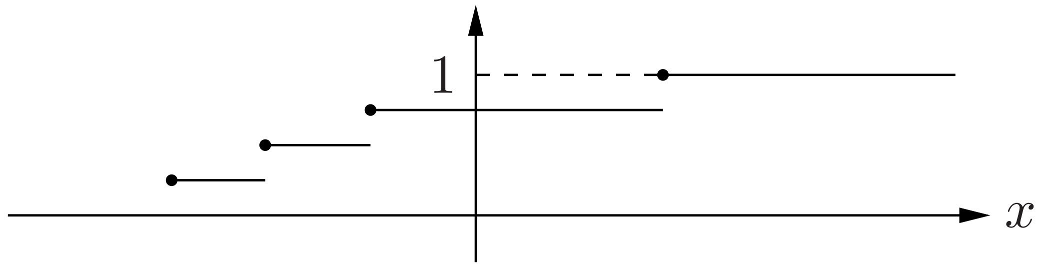


- Limits: $\lim_{x\to+\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$
- $F_X(x)$ is right continuous, i.e., $F_X(a^+) = \lim_{x\to a^+} F_X(x) = F_X(a)$
- $\mathrm{P}\{X = a\} = F_X(a) - F_X(a^-)$, where $F_X(a^-) = \lim_{x\to a^-} F_X(x)$
- For any Borel set A, $\mathrm{P}\{X \in A\}$ can be determined from $F_X(x)$

- Notation: $X \sim F_X(x)$ means that X has cdf $F_X(x)$

Probability Mass Function (PMF)

- A random variable is said to be discrete if $F_X(x)$ consists only of steps over a countable set $\mathcal{X}$



- Hence, a discrete random variable can be completely specified by the probability mass function (pmf)

$$p_X(x) = \mathrm{P}\{X = x\} \text{ for every } x \in \mathcal{X}$$

Clearly $p_X(x) \geq 0$ and $\sum_{x\in\mathcal{X}} p_X(x) = 1$

- Notation: We use $X \sim p_X(x)$ or simply $X \sim p(x)$ to mean that the discrete random variable X has pmf $p_X(x)$ or $p(x)$

- Famous discrete random variables:
  - Bernoulli: $X \sim \mathrm{Bern}(p)$ for $0 \le p \le 1$ has the pmf

$$p_X(1) = p \quad \text{and} \quad p_X(0) = 1 - p$$

  - Geometric: $X \sim \mathrm{Geom}(p)$ for $0 \le p \le 1$ has the pmf

$$p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, 3, \ldots$$

  - Binomial: $X \sim \mathrm{Binom}(n, p)$ for integer $n > 0$ and $0 \le p \le 1$ has the pmf

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots$$

  - Poisson: $X \sim \mathrm{Poisson}(\lambda)$ for $\lambda > 0$ has the pmf

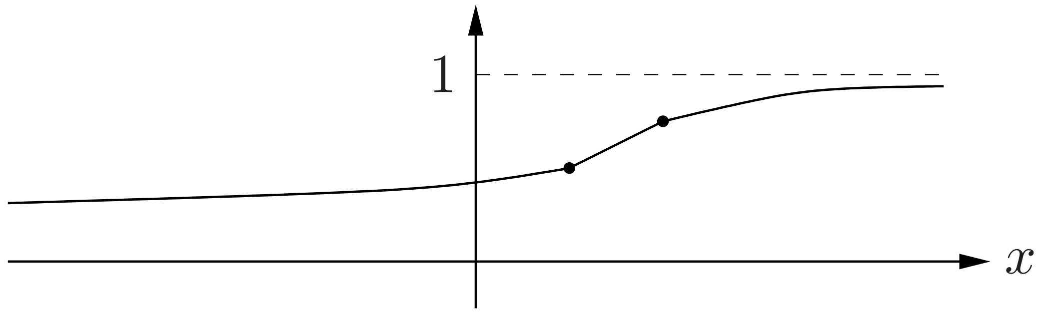
$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

  - Remark: Poisson is the limit of Binomial for $np = \lambda$ as $n \to \infty$, i.e., for every $k = 0, 1, 2, \ldots$, the $\mathrm{Binom}(n, \lambda/n)$ pmf

$$p_X(k) \to \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{as } n \to \infty$$

Probability Density Function (PDF)

- A random variable is said to be continuous if its cdf is a continuous function



- If $F_X(x)$ is continuous and differentiable (except possibly over a countable set), then X can be completely specified by a probability density function (pdf) $f_X(x)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

- If $F_X(x)$ is differentiable everywhere, then (by definition of derivative)

$$\begin{aligned} f_X(x) &= \frac{dF_X(x)}{dx} \\ &= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\mathrm{P}\{x < X \le x + \Delta x\}}{\Delta x} \end{aligned}$$

- Properties of pdf:
  - $f_X(x) \geq 0$
  - $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
  - For any event (Borel set) $A \subset \mathbf{R}$,

$$\mathrm{P}\{X \in A\} = \int_{x \in A} f_X(x)\, dx$$

    In particular,

$$\mathrm{P}\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)\, dx$$

- Important note: $f_X(x)$ should not be interpreted as the probability that $X = x$. In fact, $f_X(x)$ is not a probability measure since it can be > 1

- Notation: $X \sim f_X(x)$ means that X has pdf $f_X(x)$

- Famous continuous random variables:
  - Uniform: $X \sim \mathrm{U}[a, b]$ where $a < b$ has pdf

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

  - Exponential: $X \sim \mathrm{Exp}(\lambda)$ where $\lambda > 0$ has pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

  - Laplace: $X \sim \mathrm{Laplace}(\lambda)$ where $\lambda > 0$ has pdf

$$f_X(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$$

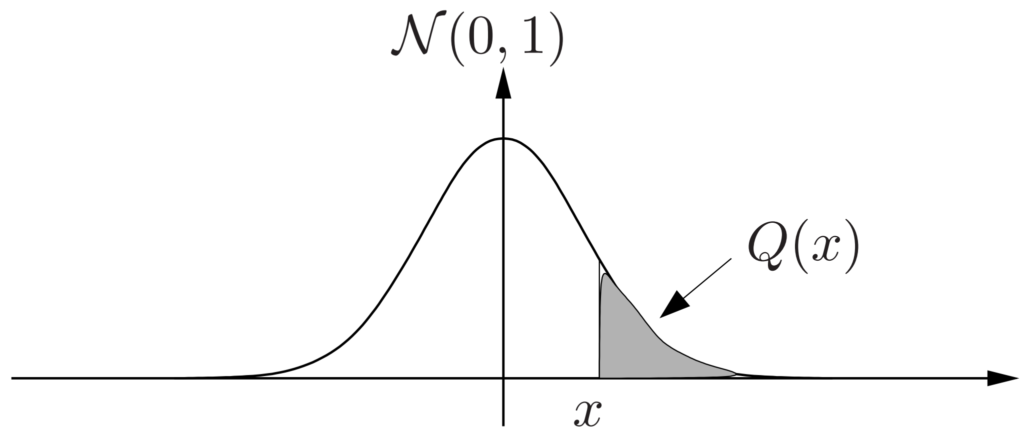
  - Gaussian: $X \sim \mathcal{N}(\mu, \sigma^2)$ with parameters μ (the mean) and σ^2 (the variance, σ is the standard deviation) has pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The cdf of the standard normal random variable $\mathcal{N}(0,1)$ is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du$$

Define the function $Q(x) = 1 - \Phi(x) = \mathrm{P}\{X > x\}$



The $Q(\cdot)$ function is used to compute $\mathrm{P}\{X > a\}$ for any Gaussian r.v. X:
Given $Y \sim \mathcal{N}(\mu, \sigma^2)$, we represent it using the standard $X \sim \mathcal{N}(0,1)$ as

$$Y = \sigma X + \mu$$

Then

$$\mathrm{P}\{Y > y\} = \mathrm{P}\left\{X > \frac{y-\mu}{\sigma}\right\} = Q\left(\frac{y-\mu}{\sigma}\right)$$

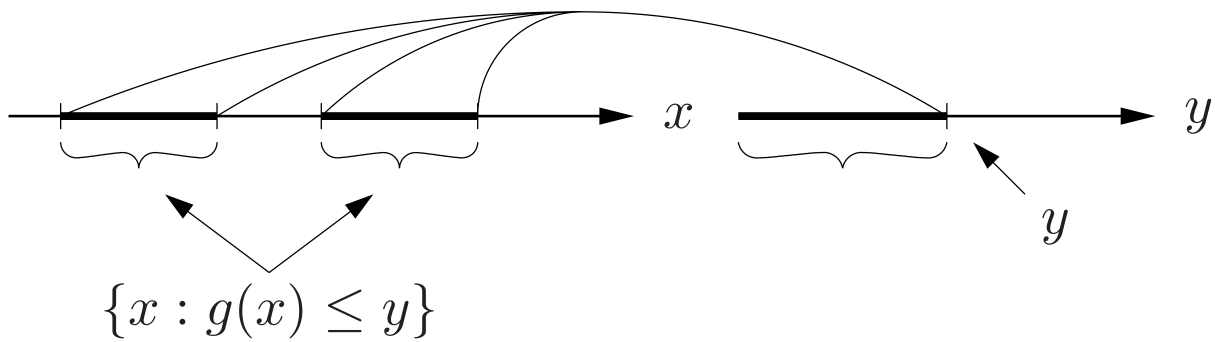
- The complementary error function is $\mathrm{erfc}(x) = 2Q(\sqrt{2}\,x)$

Functions of a Random Variable

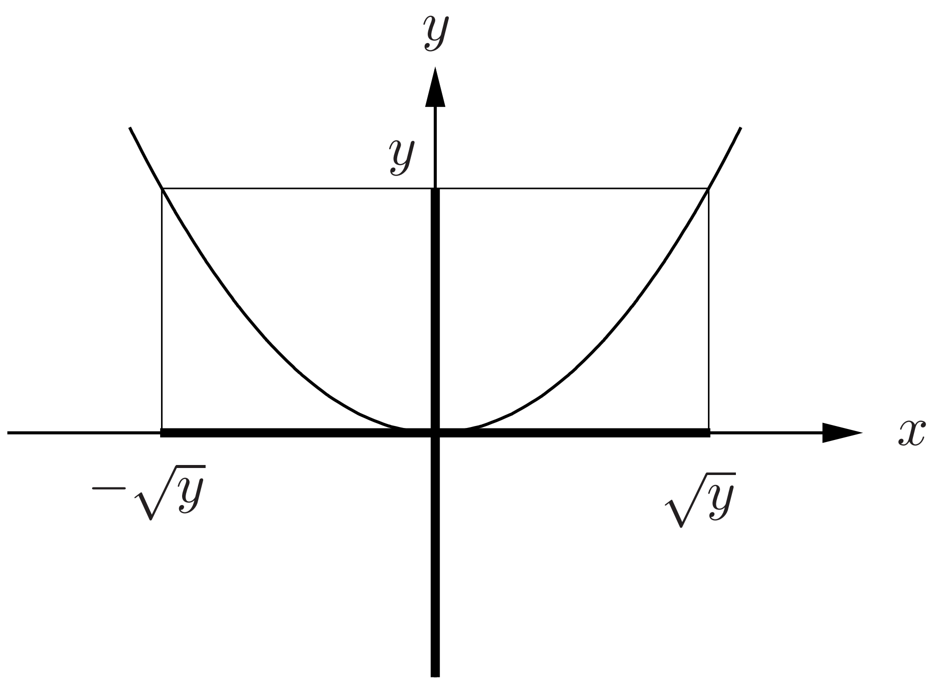
- Suppose we are given a r.v. X with known cdf $F_X(x)$ and a function $y = g(x)$. What is the cdf of the random variable $Y = g(X)$?

- We use

$$F_Y(y) \;=\; \mathrm{P}\{Y \le y\} = \mathrm{P}\{x : g(x) \le y\}$$



- Example: Square law detector. Let $X \sim F_X(x)$ and $Y = X^2$. We wish to find $F_Y(y)$



If $y < 0$, then clearly $F_Y(y) = 0$. Consider $y \geq 0$,

$$F_Y(y) = \mathrm{P}\{-\sqrt{y} < X \leq \sqrt{y}\} = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

If X is continuous with density $f_X(x)$, then

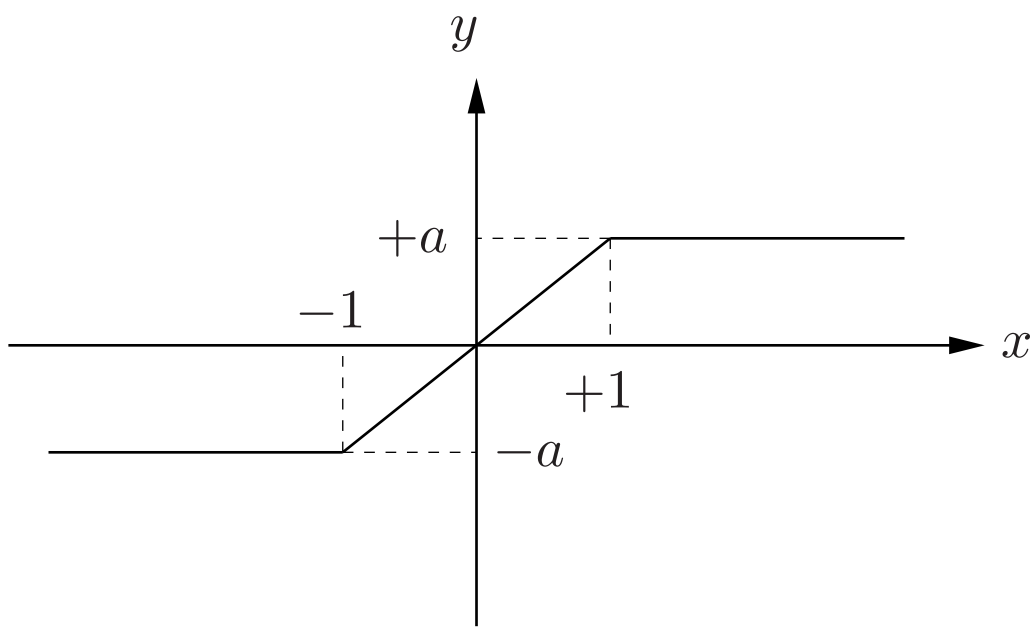
$$f_Y(y) = \frac{1}{2\sqrt{y}}\Big(f_X(+\sqrt{y}) + f_X(-\sqrt{y})\Big)$$

- Remark: In general, let $X \sim f_X(x)$ and $Y = g(X)$ be differentiable. Then

$$f_Y(y) = \sum_{i=1}^{k} \frac{f_X(x_i)}{|g'(x_i)|},$$

where $x_1, x_2, \ldots$ are the solutions of the equation $y = g(x)$ and $g'(x_i)$ is the derivative of g evaluated at x_i

- Example: Limiter. Let $X \sim \text{Laplace}(1)$, i.e., $f_X(x) = (1/2)e^{-|x|}$, and let Y be defined by the function of X shown in the figure. Find the cdf of Y



To find the cdf of Y, we consider the following cases

- $y < -a$: Here clearly $F_Y(y) = 0$
- $y = -a$: Here

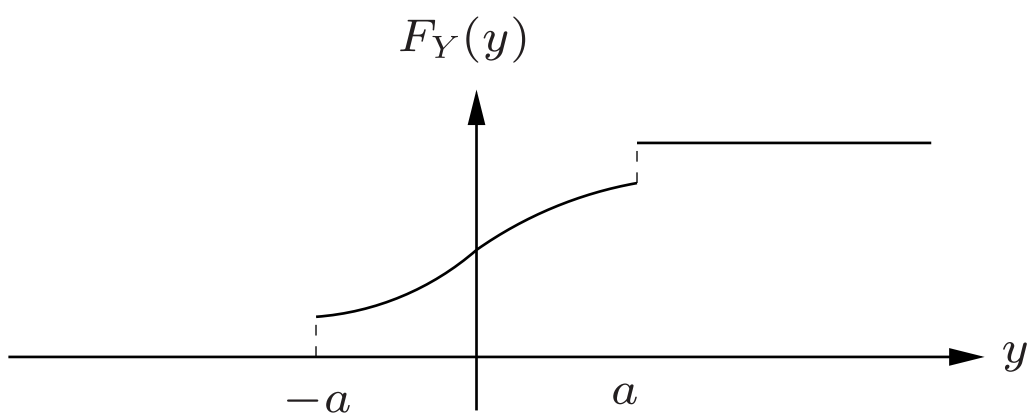
$$F_Y(-a) = F_X(-1) = \int_{-\infty}^{-1} \frac{1}{2} e^{x}\, dx = \frac{1}{2} e^{-1}$$

- $-a < y < a$: Here

$$\begin{aligned} F_Y(y) &= \mathrm{P}\{Y \le y\} \\ &= \mathrm{P}\left\{aX \le y\right\} \\ &= \mathrm{P}\left\{X \le \frac{y}{a}\right\} = F_X\left(\frac{y}{a}\right) \\ &= \tfrac{1}{2}e^{-1} + \int_{-1}^{y/a} \tfrac{1}{2}e^{-|x|}\,dx \end{aligned}$$

- $y \ge a$: Here $F_Y(y) = 1$

Combining the results, the following is a sketch of the cdf of Y



Generation of Random Variables

- Generating a r.v. with a prescribed distribution is often needed for performing simulations involving random phenomena, e.g., noise or random arrivals
- First let $X \sim F(x)$ where the cdf $F(x)$ is continuous and strictly increasing. Define $Y = F(X)$, a real-valued random variable that is a function of X

  What is the cdf of Y?

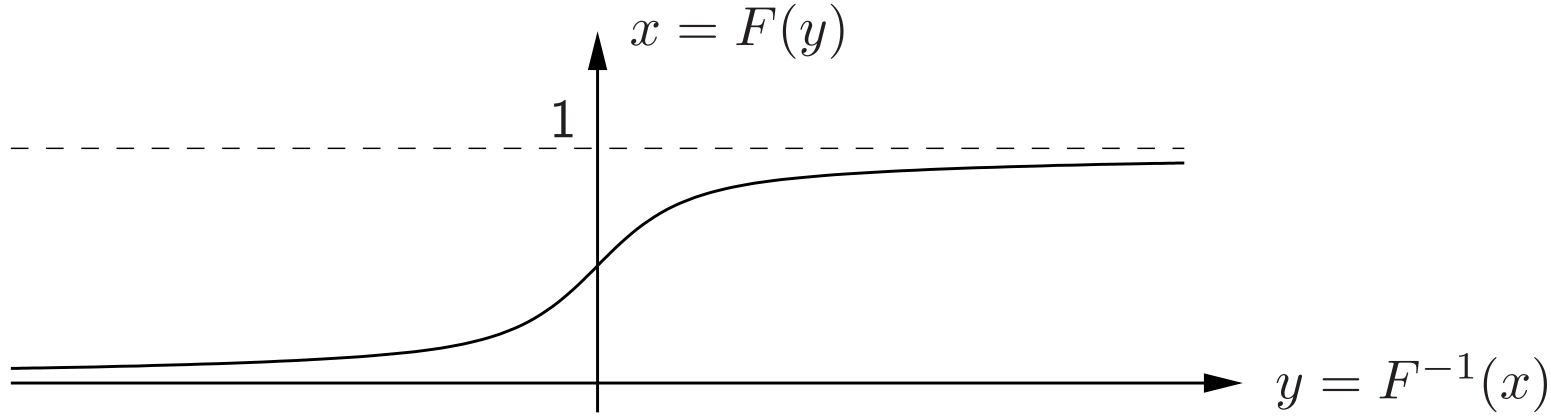
  Clearly, $F_Y(y) = 0$ for $y < 0$, and $F_Y(y) = 1$ for $y > 1$

  For $0 \le y \le 1$, note that by assumption F has an inverse F^{-1}, so

  $$F_Y(y) = \mathrm{P}\{Y \le y\} = \mathrm{P}\{F(X) \le y\} = \mathrm{P}\{X \le F^{-1}(y)\} = F(F^{-1}(y)) = y$$

  Thus $Y \sim U[\,0,1\,]$, i.e., Y is a uniformly distributed random variable
- Note: $F(x)$ does not need to be invertible. If $F(x) = a$ is constant over some interval, then the probability that X lies in this interval is zero. Without loss of generality, we can take $F^{-1}(a)$ to be the leftmost point of the interval
- Conclusion: We can generate a $\mathrm{U}[\,0,1\,]$ r.v. from any continuous r.v.

- Now, let's consider the opposite scenario where we are given $X \sim \mathrm{U}[\,0,1\,]$ (a random number generator) and wish to generate a random variable Y with prescribed cdf $F(y)$, e.g., Gaussian or exponential



- If F is continuous and strictly increasing, set $Y = F^{-1}(X)$. To show $Y \sim F(y)$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}\{Y \le y\} \\ &= \mathrm{P}\{F^{-1}(X) \le y\} \\ &= \mathrm{P}\{X \le F(y)\} \\ &= F(y)\,, \end{aligned}$$

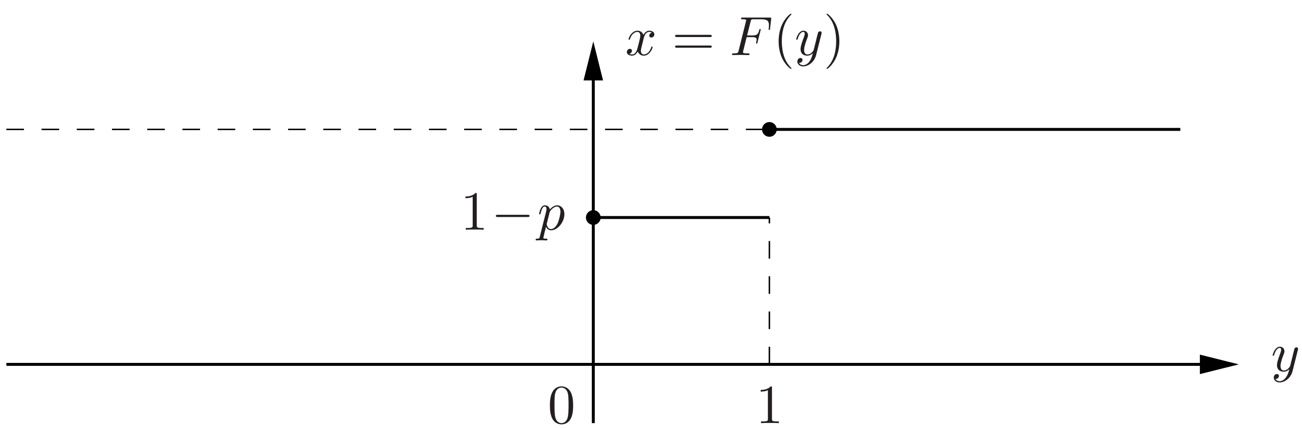
since $X \sim \mathrm{U}[\,0,1\,]$ and $0 \le F(y) \le 1$

- Example: To generate $Y \sim \mathrm{Exp}(\lambda)$, set

$$Y = -\frac{1}{\lambda}\ln(1-X)$$

- Note: F does not need to be continuous for the above to work. For example, to generate $Y \sim \mathrm{Bern}(p)$, we set

$$Y = \begin{cases} 0 & X \le 1-p \\ 1 & \text{otherwise} \end{cases}$$



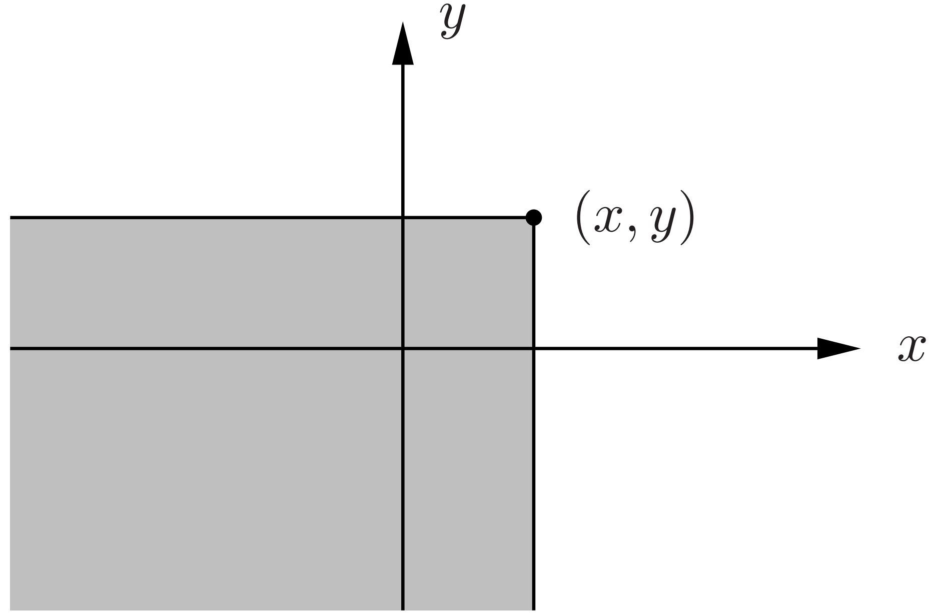
- Conclusion: We can generate a r.v. with any desired distribution from a $\mathrm{U}[0,1]$ r.v.

Jointly Distributed Random Variables

- A pair of random variables X and Y defined over the same probability space are specified by their joint cdf

$$F_{X,Y}(x,y) = \mathrm{P}\{X \leq x,\, Y \leq y\}\,, \quad x,y \in \mathbf{R}$$

$F_{X,Y}(x,y)$ is the probability of the shaded region of $\mathbf{R}^2$



- Properties of the cdf:
  - $F_{X,Y}(x,y) \geq 0$
  - If $x_1 \leq x_2$ and $y_1 \leq y_2$ then $F_{X,Y}(x_1,y_1) \leq F_{X,Y}(x_2,y_2)$
  - $\lim_{y \to -\infty} F_{X,Y}(x,y) = 0$ and $\lim_{x \to -\infty} F_{X,Y}(x,y) = 0$
  - $\lim_{y \to \infty} F_{X,Y}(x,y) = F_X(x)$ and $\lim_{x \to \infty} F_{X,Y}(x,y) = F_Y(y)$

    $F_X(x)$ and $F_Y(y)$ are the marginal cdfs of X and Y
  - $\lim_{x,y \to \infty} F_{X,Y}(x,y) = 1$
- X and Y are independent if for every x and y

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

Joint, Marginal, and Conditional PMFs

- Let X and Y be discrete random variables on the same probability space
- They are completely specified by their joint pmf:

$$p_{X,Y}(x,y) = \mathrm{P}\{X = x, Y = y\}\,, \quad x \in \mathcal{X},\, y \in \mathcal{Y}$$

By axioms of probability, $\displaystyle\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{X,Y}(x,y) = 1$

- To find $p_X(x)$, the marginal pmf of X, we use the law of total probability

$$p_X(x) = \sum_{y\in\mathcal{Y}} p(x,y)\,, \quad x \in \mathcal{X}$$

- The conditional pmf of X given $Y = y$ is defined as

$$p_{X|Y}(x|y) \; = \; \frac{p_{X,Y}(x,y)}{p_Y(y)}\,, \quad p_Y(y) \neq 0,\, x \in \mathcal{X}$$

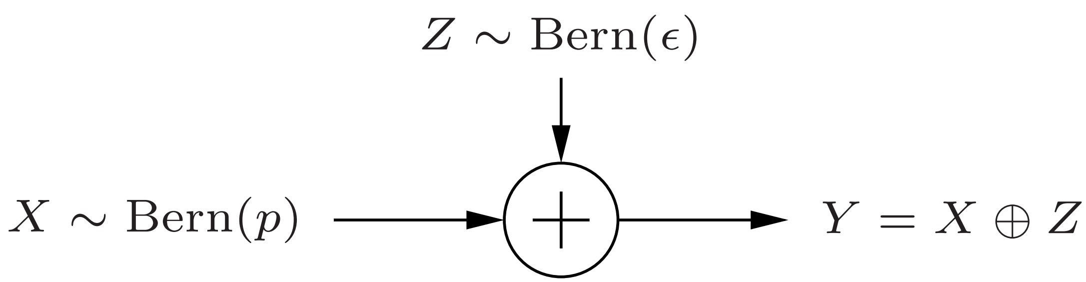
- Chain rule: $p_{X,Y}(x,y) = p_X(x)p_{Y|X}(y|x) = p_Y(y)p_{X|Y}(x|y)$

- Independence: X and Y are said to be independent if for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$,

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)\,,$$

which is equivalent to $p_{X|Y}(x|y) = p_X(x)$ for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $p_Y(y) \neq 0$

Example (Binary Symmetric Channel):



X and Z are independent

Joint, Marginal, and Conditional PDF

- X and Y are jointly continuous random variables if their joint cdf is continuous in both x and y

  In this case, we can define their joint pdf, provided that it exists, as the function $f_{X,Y}(x,y)$ such that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, du\, dv\,, \quad x, y \in \mathbf{R}$$

- If $F_{X,Y}(x,y)$ is differentiable in x and y, then

$$f_{X,Y}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \lim_{\Delta x, \Delta y \to 0} \frac{\mathrm{P}\{x < X \le x + \Delta x,\ y < Y \le y + \Delta y\}}{\Delta x \Delta y}$$

- Properties of $f_{X,Y}(x,y)$:
  - $f_{X,Y}(x,y) \ge 0$
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$

- The marginal pdf of X can be obtained from the joint pdf via the law of total probability:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$

- X and Y are independent iff $f_{X,Y}(x, y) = f_X(x) f_Y(y)$ for every x, y

- Conditional cdf and pdf: Let X and Y be continuous random variables with joint pdf $f_{X,Y}(x, y)$. We wish to define $F_{Y|X}(y \,|\, X = x) = \mathrm{P}\{Y \le y \,|\, X = x\}$

  We cannot define the above conditional probability as

$$\frac{\mathrm{P}\{Y \le y,\, X = x\}}{\mathrm{P}\{X = x\}}$$

  because both numerator and denominator are equal to zero. Instead, we define conditional probability for continuous random variables as a limit

$$\begin{aligned} F_{Y|X}(y|x) &= \lim_{\Delta x \to 0} \mathrm{P}\{Y \le y \,|\, x < X \le x + \Delta x\} \\ &= \lim_{\Delta x \to 0} \frac{\mathrm{P}\{Y \le y,\, x < X \le x + \Delta x\}}{\mathrm{P}\{x < X \le x + \Delta x\}} \\ &= \lim_{\Delta x \to 0} \frac{\int_{-\infty}^{y} f_{X,Y}(x, u)\, du\, \Delta x}{f_X(x) \Delta x} = \int_{-\infty}^{y} \frac{f_{X,Y}(x, u)}{f_X(x)}\, du \end{aligned}$$

- We then define the conditional pdf in the usual way as

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \qquad \text{if } f_X(x) \neq 0$$

- Thus

$$F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(u|x)\,du$$

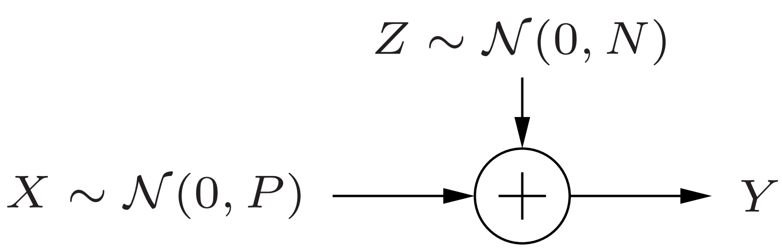
which shows that $f_{Y|X}(y|x)$ is a pdf for Y given $X = x$, i.e.,

$$Y \mid \{X = x\} \sim f_{Y|X}(y|x)$$

- Chain rule: $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x)$

- Independence: X and Y are independent if $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for every (x,y), or equivalently, $f_{Y|X}(y|x) = f_Y(y)$

- Example (Additive Gaussian channel):



X and Z are independent

One Discrete and One Continuous Random Variables

- Let Θ be a discrete random variable with pmf $p_\Theta(\theta)$
- For each $\Theta = \theta$ with $p_\Theta(\theta) \neq 0$, let Y be a continuous random variable, i.e., $F_{Y|\Theta}(y|\theta)$ is continuous for all θ. We define $f_{Y|\Theta}(y|\theta)$ in the usual way
- The conditional pmf of Θ given y can be defined as a limit

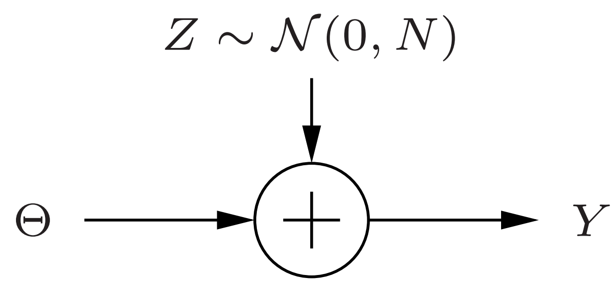
$$\begin{aligned} p_{\Theta|Y}(\theta \,|\, y) &= \lim_{\Delta y \to 0} \frac{\mathrm{P}\{\Theta = \theta,\, y < Y \leq y + \Delta y\}}{\mathrm{P}\{y < Y \leq y + \Delta y\}} \\ &= \lim_{\Delta y \to 0} \frac{p_\Theta(\theta) f_{Y|\Theta}(y|\theta)\Delta y}{f_Y(y)\Delta y} = \frac{f_{Y|\Theta}(y|\theta) p_\Theta(\theta)}{f_Y(y)} \end{aligned}$$

  This leads to the Bayes rule:

$$p_{\Theta|Y}(\theta \,|\, y) = \frac{f_{Y|\Theta}(y|\theta)}{\sum_{\theta'} p_\Theta(\theta') f_{Y|\Theta}(y|\theta')} p_\Theta(\theta)$$

- Example: Additive Gaussian Noise Channel

  Consider the following noisy channel:



  The signal transmitted is a binary random variable Θ:

$$\Theta = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

  The received signal, also called the observation, is $Y = \Theta + Z$, where Θ and Z are independent

  Given $Y = y$ is received (observed), find $p_{\Theta|Y}(\theta|y)$, the a posteriori pmf of Θ

- In some cases we are given $f_Y(y)$ and $p_{\Theta|Y}(\theta|y)$ for every y
- We can find the a posteriori pdf of Y using the Bayes rule:

$$f_{Y|\Theta}(y|\theta) = \frac{p_{\Theta|Y}(\theta|y)}{\int f_Y(y')p_{\Theta|Y}(\theta|y')dy'} f_Y(y)$$

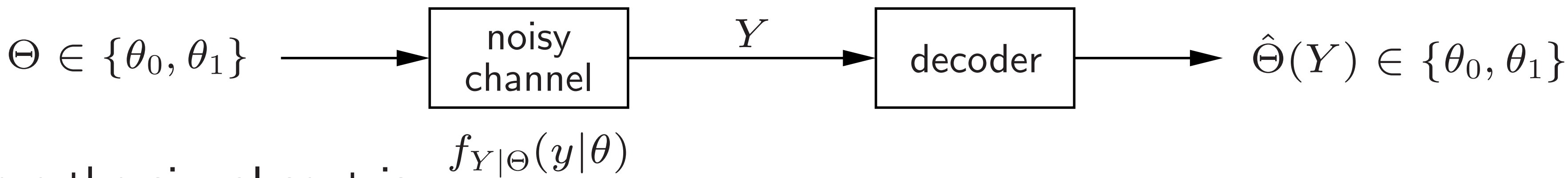
- Example: Coin with random bias

  Consider a coin with random bias $P \sim f_P(p)$. Flip the coin and let $X = 1$ if the outcome is heads and $X = 0$ if the outcome is tails

  Given that $X = 1$ (i.e., outcome is heads), find $f_{P|X}(p|1)$, the a posteriori pdf of P

Scalar Detection

- Consider the following signal processing problem:



where the signal sent is

$$\Theta = \begin{cases} \theta_0 & \text{with probability } p \\ \theta_1 & \text{with probability } 1-p \end{cases}$$

and the observation (received signal) is

$$Y \,|\, \{\Theta = \theta\} \sim f_{Y|\Theta}(y\,|\,\theta)\,, \quad \theta \in \{\theta_0, \theta_1\}$$

- We wish to find the estimate $\hat{\Theta}(Y)$ (i.e., design the decoder) that minimizes the probability of error:

$$\begin{aligned} P_e &\stackrel{\triangle}{=} \mathrm{P}\{\hat{\Theta} \neq \Theta\} = \mathrm{P}\{\Theta = \theta_0,\ \hat{\Theta} = \theta_1\} + \mathrm{P}\{\Theta = \theta_1,\ \hat{\Theta} = \theta_0\} \\ &= \mathrm{P}\{\Theta = \theta_0\}\mathrm{P}\{\hat{\Theta} = \theta_1 \,|\, \Theta = \theta_0\} + \mathrm{P}\{\Theta = \theta_1\}\mathrm{P}\{\hat{\Theta} = \theta_0 \,|\, \Theta = \theta_1\} \end{aligned}$$

- We define the maximum a posteriori probability (MAP) decoder as

$$\hat{\Theta}(y) = \begin{cases} \theta_0 & \text{if } p_{\Theta|Y}(\theta_0|y) > p_{\Theta|Y}(\theta_1|y) \\ \theta_1 & \text{otherwise} \end{cases}$$

- The MAP decoding rule minimizes P_e, since

$$\begin{aligned} \min_{\hat{\Theta}} P_e &= 1 - \max_{\hat{\Theta}} \mathrm{P}\{\hat{\Theta}(Y) = \Theta\} \\ &= 1 - \max_{\hat{\Theta}} \int_{-\infty}^{\infty} f_Y(y) \mathrm{P}\{\Theta = \hat{\Theta}(y) \,|\, Y = y\}\, dy \\ &= 1 - \int_{-\infty}^{\infty} f_Y(y) \max_{\hat{\Theta}(y)} \mathrm{P}\{\Theta = \hat{\Theta}(y) \,|\, Y = y\}\, dy \end{aligned}$$

  and the probability of error is minimized if we pick the largest $p_{\Theta|Y}(\hat{\Theta}(y)|y)$ for every y, which is precisely the MAP decoder

- If $p = \frac{1}{2}$, i.e., equally likely signals, using Bayes rule, the MAP decoder reduces to the maximum likelihood (ML) decoder

$$\hat{\Theta}(y) = \begin{cases} \theta_0 & \text{if } f_{Y|\Theta}(y|\theta_0) > f_{Y|\Theta}(y|\theta_1) \\ \theta_1 & \text{otherwise} \end{cases}$$

Additive Gaussian Noise Channel

- Consider the additive Gaussian noise channel with signal

$$\Theta = \begin{cases} +\sqrt{P} & \text{with probability } \frac{1}{2} \\ -\sqrt{P} & \text{with probability } \frac{1}{2} \end{cases}$$

  noise $Z \sim \mathcal{N}(0, N)$ (Θ and Z are independent), and output $Y = \Theta + Z$

- The MAP decoder is

$$\hat{\Theta}(y) = \begin{cases} +\sqrt{P} & \text{if } \dfrac{\mathrm{P}\{\Theta = +\sqrt{P} \,|\, Y = y\}}{\mathrm{P}\{\Theta = -\sqrt{P} \,|\, Y = y\}} > 1 \\ -\sqrt{P} & \text{otherwise} \end{cases}$$

  Since the two signals are equally likely, the MAP decoding rule reduces to the ML decoding rule

$$\hat{\Theta}(y) = \begin{cases} +\sqrt{P} & \text{if } \dfrac{f_{Y|\Theta}(y \,|+ \sqrt{P})}{f_{Y|\Theta}(y \,|- \sqrt{P})} > 1 \\ -\sqrt{P} & \text{otherwise} \end{cases}$$

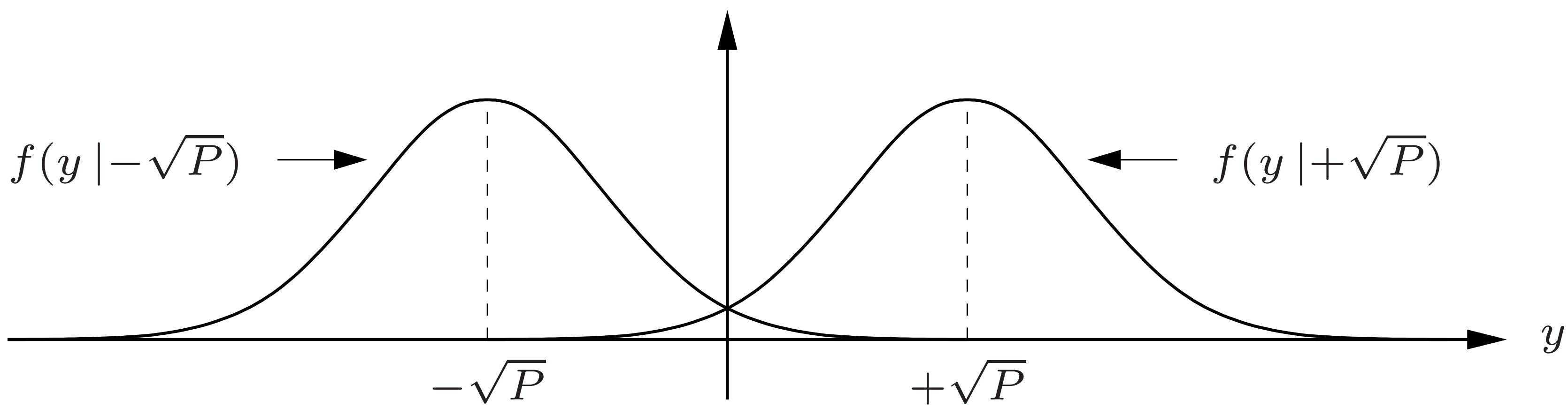
- Using the Gaussian pdf, the ML decoder reduces to the minimum distance decoder

$$\hat{\Theta}(y) = \begin{cases} +\sqrt{P} & (y-\sqrt{P})^2 < (y-(-\sqrt{P}))^2 \\ -\sqrt{P} & \text{otherwise} \end{cases}$$

From the figure, this simplifies to

$$\hat{\Theta}(y) = \begin{cases} +\sqrt{P} & y > 0 \\ -\sqrt{P} & y < 0 \end{cases}$$

Note: The decision when $y = 0$ is arbitrary



- Now to find the minimum probability of error, consider

$$\begin{aligned}
P_e &= \mathrm{P}\{\hat{\Theta}(Y) \neq \Theta\} \\
&= \mathrm{P}\{\Theta = \sqrt{P}\}\mathrm{P}\{\hat{\Theta}(Y) = -\sqrt{P} \,|\, \Theta = \sqrt{P}\} + \\
&\qquad \mathrm{P}\{\Theta = -\sqrt{P}\}\mathrm{P}\{\hat{\Theta}(Y) = \sqrt{P} \,|\, \Theta = -\sqrt{P}\} \\
&= \tfrac{1}{2}\mathrm{P}\{Y \leq 0 \,|\, \Theta = \sqrt{P}\} + \tfrac{1}{2}\mathrm{P}\{Y > 0 \,|\, \Theta = -\sqrt{P}\} \\
&= \tfrac{1}{2}\mathrm{P}\{Z \leq -\sqrt{P}\} + \tfrac{1}{2}\mathrm{P}\{Z > \sqrt{P}\} \\
&= Q\left(\sqrt{\frac{P}{N}}\right) = Q\left(\sqrt{\mathrm{SNR}}\right)
\end{aligned}$$

The probability of error is a decreasing function of P/N, the signal-to-noise ratio (SNR)