On the Capacity of AWGN Relay Channels with Linear Relaying Functions

Sina Zahedi, Mehdi Mohseni and Abbas El Gamal

Information Systems Laboratory
Stanford University

July 1, 2004
Relay Channel

- Discrete-memoryless relay channel [van der Meulen’ 71]

At each time $i$, relay transmission $x'_i = f_i(y'_1, y'_2, \ldots, y'_{i-1})$

- Max-Flow Min-Cut upper bound

$$C \leq \max_{p(x,x')} \min \{I(X, X'; Y), I(X; Y, Y'|X')\}$$

- Block-Markov encoding lower bound

$$C \geq \max_{p(x,x')} \min \{I(X, X'; Y), I(X; Y|X')\}$$

- Capacity is not known, in general
**AWGN Relay Channel**

- **General AWGN relay channel**

  \[ Z_1 \sim \mathcal{N}(0, N) \]

  ![Diagram of General AWGN relay channel]

- **Frequency-Division AWGN (FD-AWGN) relay channel**

  \[ Z_1 \sim \mathcal{N}(0, N) \quad Z_R \sim \mathcal{N}(0, N) \]

  ![Diagram of Frequency-Division AWGN relay channel]

- Assume individual average power constraint \( P \) on \( X \) and \( X' \)
# Upper and Lower Bounds on Capacity

<table>
<thead>
<tr>
<th></th>
<th>Block-Markov Lower Bound ((a \geq 1))</th>
<th>Max-Flow Min-Cut Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>General AWGN</td>
<td>(a^2 \geq 1 + b^2)</td>
<td>(a^2 \geq b^2)</td>
</tr>
<tr>
<td></td>
<td>(a^2 &lt; 1 + b^2)</td>
<td>(a^2 &lt; b^2)</td>
</tr>
<tr>
<td>FD-AWGN</td>
<td>(a^2 \geq 1 + b^2 + \frac{b^2 P}{N})</td>
<td>(a^2 \geq b^2 + \frac{b^2 P}{N})</td>
</tr>
<tr>
<td></td>
<td>(a^2 &lt; 1 + b^2 + \frac{b^2 P}{N})</td>
<td>(a^2 &lt; b^2 + \frac{b^2 P}{N})</td>
</tr>
</tbody>
</table>

- General AWGN channel model: Capacity is not known for any \(a, b > 0\)
- FD-AWGN channel model: If \(a^2 \geq 1 + b^2 + \frac{b^2 P}{N}\) capacity is given by \(C \left( \frac{P}{N} \left( 1 + b^2 + \frac{b^2 P}{N} \right) \right) \)
For $a < 1$, the best lower bound obtained by the block-Markov encoding scheme is $C(P/N)$.

Are there any simple encoding schemes achieving rates higher than $C(P/N)$?

Consider the case where the relay node can only transmit a linear combination of the past received symbols, i.e.,

$$x_{i+1}' = \sum_{j=1}^{i} d_{ij} y_j'$$

Using vector notation

$$X' = DY',$$

where $D = [d_{ij}]$ is a lower triangular matrix.

Let $C(l)$ be the capacity with linear relaying functions.
Consider the following sub-optimal linear relaying scheme with block length 2

\[ X_1 \sim \mathcal{N}(0, 2\alpha P), \quad \text{and} \quad X_2 = \sqrt{\frac{1-\alpha}{\alpha}} X_1 \quad \text{and} \quad X'_2 = dY'_1 \quad \text{where} \quad d \quad \text{is chosen to satisfy the relay power constraint} \]

The achievable rate is

\[
R = \frac{1}{2} I(X_1, X_2; Y_1, Y_2) \\
= \max_{0 \leq \alpha \leq 1} \frac{1}{2} C \left( \frac{2\alpha P}{N} \left( 1 + \frac{\left( \frac{\sqrt{(1 - \alpha)} / \alpha + abd}{1 + b^2d^2} \right)^2}{\sqrt{(1 - \alpha)} / \alpha + abd} \right) \right) \leq C^{(l)}
\]
Example $a = 1$ and $b = 2$
Capacity with Linear Relaying

- Let $C^{(l)}_k = \sup_{X^k, D} I(X^k; Y^k)$ subject to $\sum_{i=1}^k E(X_i^2) \leq kP$ and $\max_y \sum_{i=1}^k x_i^2 \leq kP$

- Linear capacity can be expressed as

$$C^{(l)} = \sup_k \frac{1}{k} C^{(l)}_k = \lim_{k \to \infty} \frac{1}{k} C^{(l)}_k$$

- For the general AWGN relay channel, problem can be reduced to

$$C^{(l)} = \lim_{k \to \infty} \max \frac{1}{\Sigma_x, D} \log \frac{|(I_k + abD)^T \Sigma_x (I_k + abD)^T + N(I_k + b^2DD^T)|}{|N(I_k + b^2DD^T)|}$$

Subject to

$\Sigma_x \succeq 0$

$\text{tr}(\Sigma_x) \leq kP$

$\text{tr}(a^2\Sigma_x D^T D + N D^T D) \leq kP$

$d_{ij} = 0 \text{ for } j \geq i$

- Sequence of non-convex problems (open problem)

- Surprisingly, we can find “single-letter” characterization for the FD-AWGN relay channel!
Example: Consider the following transmission scheme

\[ X \xrightarrow{a} Y' \xrightarrow{b} Y_R \]

- \( X_1, X_2, \ldots \) are i.i.d. \( \sim \mathcal{N}(0, P) \), and \( x'_i = d y'_i \) where coefficient \( d \) is chosen to satisfy the relay power constraint.

- The achievable rate for this scheme is

\[
C_1^{(l)} = C \left( \frac{P}{N} \left( 1 + \frac{a^2 b^2 P}{(a^2 + b^2)P + N} \right) \right)
\]
\textbf{Main Result}

- $C_1^{(l)}$ is not a concave function of $P$ and can be increased by time-sharing.

\begin{itemize}
  \item \textbf{Theorem}: Capacity of FD-AWGN relay channel with linear relaying is given by
  \[
  C^{(l)} = \lim_{k \to \infty} \frac{1}{k} C_k^{(l)} = \max_{0 < \alpha \leq 1} \alpha C \left( \frac{P}{\alpha N} \left( 1 + \frac{a^2 b^2 P}{(a^2 + b^2) P + \alpha N} \right) \right)
  \]
  \item Note that $C^{(l)} = \max_{0 < \alpha \leq 1} \alpha C_1^{(l)} \left( \frac{P}{\alpha} \right)$, i.e., a concavified version of $C_1^{(l)}$.
\end{itemize}
Proof

- Finding $C_{k}^{(l)}$ reduces to

Maximize \[ \frac{1}{2k} \log_2 \left| \begin{bmatrix} \Sigma_x + NI & ab\Sigma_x D^T \\ abD\Sigma_x & a^2b^2D\Sigma_x D^T + N(I + b^2DD^T) \end{bmatrix} \right| \]

Subject to
\[
\begin{align*}
\Sigma_x & \succeq 0 \\
\text{tr}(\Sigma_x) & \leq kP \\
\text{tr}(a^2\Sigma_x D^TD + ND^TD) & \leq kP \\
d_{ij} & = 0 \text{ for } j > i
\end{align*}
\]

- $\Sigma_x = E(XX^T)$ and $D$ are the optimization variables
Proof (continued)

- Objective function can be expressed as

\[
\frac{1}{2k} \log_2 \left| I + G \Sigma_x G^T \right|
\]

where

\[
G = \frac{1}{\sqrt{N}} \begin{bmatrix}
I \\
ab \sqrt{I + b^2 D D^T}^{-1} D
\end{bmatrix}
\]

- Finding \( \Sigma_x \) for a fixed \( D \) is a convex problem (similar to waterfilling); however, finding \( D \) for a fixed \( \Sigma_x \) is a non-convex problem

- Let \( U \Theta V^T \) be the singular value decomposition of \( G \), the optimal \( \Sigma_x \) is of the form \( \Sigma_x = V \Psi V^T \)

- Key observation: \( G \) and \( D \) share the same set of right eigenvectors, hence \( D^T D = V \Lambda V^T \)
Proof (continued)

• The objective function can be simplified to

\[
\frac{1}{2k} \log_2 \left| I + \frac{1}{N} \Psi(I + a^2 b^2 \Lambda(I + b^2 \Lambda)^{-1}) \right|
\]

• This doesn’t depend on \( V \), so we set \( V = I \), then \( \Sigma_x = \Psi \) and \( D^2 = \Lambda \)

• The problem can be simplified to

Maximize

\[
\frac{1}{2k} \log_2 \left( \prod_{i=1}^{k} \left( 1 + \frac{1}{N} \sigma_i \left( 1 + \frac{a^2 b^2 d_i^2}{1 + b^2 d_i^2} \right) \right) \right)
\]

Subject to \( \sigma_i \geq 0 \), for \( i = 1, 2, \ldots, k \), \( \sum_{i=1}^{k} \sigma_i \leq kP \), and

\[
\sum_{i=1}^{k} d_i^2 (a^2 \sigma_i + N) \leq kP
\]
Proof (continued)

- This again is a non-convex optimization problem
- Using a sequence of inequalities, it can be shown that
  \[
  \frac{1}{2k} \log_2 \prod_{i=1}^{k} \left( 1 + \frac{1}{N} \sigma_i \left( 1 + \frac{a^2 b^2 d_i^2}{1 + b^2 d_i^2} \right) \right) \leq \max_{0 \leq n \leq k} \frac{n}{2k} \log_2 \left( 1 + \frac{kP}{nN} \left( 1 + \frac{a^2 b^2 kP}{(a^2 + b^2)kP + nN} \right) \right)
  \]
- The upper bound is achievable by setting
  \[
  \sigma_1 = \ldots = \sigma_n = \frac{kp}{n}, \sigma_{n+1} = \ldots = \sigma_k = 0
  \]
  and
  \[
  d_1 = \ldots = d_n = \frac{kP}{\sqrt{a^2kP + nN}}, d_{n+1} = \ldots = d_k = 0
  \]
- For each block length \(k\), the following rate is achievable
  \[
  \frac{1}{k} C_k^{(l)} = \max_{1 \leq n \leq k} \frac{n}{k} C \left( \frac{kP}{nN} \left( 1 + \frac{a^2 b^2 kP}{(a^2 + b^2)kP + nN} \right) \right)
  \]
  which completes the proof
Example $a = 1$ and $b = 2$
Example $a = 1$ and $b = 5$

- As $b \to \infty$, $C^{(l)}$ approaches the Max-Flow Min-Cut upper bound
Conclusion

- Discussed bounds for the general and frequency-division AWGN relay channels
- Investigated achievable rates for linear relaying functions:
  - For the general AWGN relay channel: can do better than other schemes, optimal rate not known
  - For the frequency-division AWGN relay channel: optimal rate found, use first-order Markov and time-sharing