Minimum Energy Communication Over AWGN Relay Channels

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Motivation

• Growing interest in ad hoc (e.g., sensor) wireless networks:
  – Nodes can communicate through other nodes (relays)
• Relaying increases throughput (capacity) (Gupta, Kumar’ 2000) by reducing transmission range and thus reducing interference
• Reducing transmission range also reduces total transmission energy – key concern in mobile wireless networks
• What is the smallest transmission energy-per-bit at which reliable communication in an ad hoc network is possible?
• We provide partial answer to this question for the AWGN relay channel, the simplest example of ad hoc network
Relay Channel – Brief History

- Discrete-memoryless relay channel (van der Meulen’ 71)

- Capacity is not known, in general

- Capacity is known for *Physically degraded* (block-Markov scheme) and *reversely degraded* (Cover, El Gamal’ 79), and *semi-deterministic* relay channel (El Gamal, Aref’ 81)


- Max-Flow Min-Cut upper bound (Cover, El Gamal’ 79, El Gamal’ 81)
AWGN Relay Channel

- $a > 0$ and $b > 0$ are relative channel gains, and $Z_1$ and $Z$ are independent $\mathcal{N}(0, N)$

- A $(2^n R, n)$ relay channel code consists of:
  - A message set $\mathcal{W} = \{1, 2, \ldots, 2^n R\}$,
  - A codebook $\{x_1, x_2, \ldots x_{2^n R}\}$ for sender $X$,
  - A set of relay functions such that $x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1i-1})$, $1 \leq i \leq n$, and
  - A decoding function $\hat{W}(y) \in \mathcal{W}$

- Achievability of $R$ is defined in the usual way
Capacity With Average Power Constraints

- Assume average power constraint $P$ on $X$, and $\gamma P$, $\gamma \geq 0$, on $X_1$
- Capacity with average power constraints $C(P, \gamma P)$
- Capacity is not known for any $a, b > 0$
- If the relay not used: $C(P, \gamma P) > \frac{1}{2} \log_2 \left( 1 + \frac{P}{N} \right) \overset{\triangle}{=} C \left( \frac{P}{N} \right)$
- Using block Markov scheme (Cover, El Gamal’ 79) for $a > 1$:
  \[
  C(P, \gamma P) \geq \max_{0 \leq \rho \leq 1} \min \left\{ C \left( \frac{(1 + b^2 \gamma + 2b \rho \sqrt{\gamma})P}{N} \right), C \left( \frac{a^2(1 - \rho^2)P}{N} \right) \right\}
  \]
- Using the Max-flow Min-cut bound (Cover, El Gamal’ 79):
  \[
  C(P, \gamma P) \leq \max_{0 \leq \rho \leq 1} \min \left\{ C \left( \frac{(1 + b^2 \gamma + 2b \rho \sqrt{\gamma})P}{N} \right), C \left( \frac{(1 + a^2)(1 - \rho^2)P}{N} \right) \right\}
  \]
Minimum Energy-Per-Bit

• Consider a \( (2^{nR}, n) \) code

• The energy for codeword \( x_k \) is

\[
E_{k}^{(n)} = \sum_{i=1}^{n} x_{ik}^{2},
\]

and the maximum relay energy is

\[
E_{r}^{(n)} = \max_{y_1} \sum_{i=1}^{n} x_{1i}^{2}
\]

• Define the energy-per-bit for the code as

\[
E_{n} = \frac{1}{nR} \left( \max_k E_{k}^{(n)} + E_{r}^{(n)} \right)
\]

• Energy-per-bit \( E \) is achievable if there is a sequence of codes with probability of error \( \rightarrow 0 \) and \( \lim \sup E_{n} \leq E \)

• The minimum energy-per-bit \( E_b \) is infimum of the set of achievable \( E \)
  
  – Single number that depends only on \( a, b, \) and \( N \)
Relationship Between $\mathcal{E}_b$ and $C(P, \gamma P)$

- We can express capacity for the relay channel as
  \[
  C(P, \gamma P) = \sup_k C_k(P, \gamma P),
  \]
  where
  \[
  C_k(P, \gamma P) = \frac{1}{k} \sup_{P, X^k, \{f_i\}_{i=1}^k: \sum_i^k E(X_i^2) \leq kP, \max_{y_1} \left(\sum_i^k x_i^2\right) \leq k\gamma P} I(X^k; Y^k)
  \]

- This can be used to show that
  \[
  \mathcal{E}_b = \inf_{P, \gamma \geq 0} \frac{(1 + \gamma)P}{C(P, \gamma P)}
  \]

- It can be shown that $C(P, \gamma P)$ is concave and strictly increasing in $P$, thus $\frac{(1+\gamma)P}{C(P,\gamma P)}$ is non-decreasing in $P$, and
  \[
  \mathcal{E}_b = \inf_{\gamma \geq 0} \lim_{P \to 0} \frac{(1 + \gamma)P}{C(P, \gamma P)}
  \]
  - Is $\mathcal{E}_b$ is a limit: Is it easier to find than $C(P, \gamma P)$?
  - Unfortunately, the answer appears to be NO!
Bounds on $\mathcal{E}_b$

- Using the relationship between $\mathcal{E}_b$ and $C(P, \gamma P)$ and the bounds on capacity we obtain

$$\frac{1 + a^2 + b^2}{(1 + a^2)(1 + b^2)} \leq \frac{\mathcal{E}_b}{2N \ln 2} \leq \min \left\{ 1, \frac{a^2 + b^2}{a^2(1 + b^2)} \right\}$$

- Lower bound obtained using Max-flow Min-Cut upper bound
- Upper bounds obtained using direct transmission for $a \leq 1$ and block Markov bound for $a > 1$

- The bounds are not tight for any $a, b > 0$

- **Example:** Assume relay is half way between sender and receiver ($a = b = 4$)

  - Relay not used: Minimum energy-per-bit is $2N \ln 2$
  - Block Markov coding upper bound gives $0.235N \ln 2$
  - Max-Flow Min-Cut lower bound gives $0.228N \ln 2$
Ratio of Upper to Lower Bounds $\leq 2$

![Graph showing the ratio of upper to lower bounds on $\mathcal{E}_0$ as a function of $a$. The graph is labeled with $b=10$ and various values of $\alpha$. The ratio is shown to be less than or equal to 2 for all values of $a$.](image-url)
Improving the Upper Bound for $a \approx 1$

- Using the quantization-side information coding scheme in (Cover, El Gamal '79), any rate
  \[ R = \min \{ I(X, X_1; Y) - I(Y_1; \hat{Y}_1|X, X_1), I(X; Y, \hat{Y}_1|X_1) \}, \]
  where \( p(x, x_1, y_1, \hat{y}_1, y) = p(x)p(x_1)p(y_1|x)p(\hat{y}_1|y_1)p(y|x, x_1) \), is achievable

- Assume \( X, X_1 \) to be Gaussian and \( \hat{Y}_1 = \alpha(Y_1 + Z') \), where \( Z' \) is Gaussian independent of \( X, X_1, Z, \) and \( Z_1 \), we obtain
  \[ R(P, \gamma P) = \left( \frac{P}{N} \left( 1 + \frac{a^2 b^2 \gamma P}{P(1 + a^2 + b^2 \gamma) + N} \right) \right) \]

- As \( b \to \infty \), this bound is tight (Gastpar' 2002)

- Using this bound, we obtain
  \[ \mathcal{E}_b \leq \min_{\gamma \geq 0} \min_P \left( \frac{1 + \gamma}{R(P, \gamma P)} \right) \]

  - \( R(P, \gamma P) \) is not concave and \( \min_P \) attained at \( P \geq 0 \)
  - \( \lim_{P \to 0} \) is \( 2N \ln 2 \)!
The Max-flow Min-cut upper bound, and block Markov and direct transmission lower bounds on capacity yield

\[
\min \left\{ \frac{a^2 + b^2}{(1 + a^2)b^2}, 1 \right\} \leq \frac{\mathcal{E}_b^A}{2N \ln 2} \leq \begin{cases} \frac{a^2 + b^2 - 1}{a^2 b^2}, & \text{if } a, b > 1 \\ 1, & \text{if } a \leq 1 \text{ or } b \leq 1 \end{cases}
\]

- Ratio of bounds \( \leq 2 \)
- For \( b \leq 1 \), bounds are tight and \( \mathcal{E}_b^A = 2N \ln 2 \) (independent of \( a \))
- Bounds can be improved for \( a \approx 1 \) and large \( b \) using quantization scheme
Capacity and minimum energy-per-bit are given by

$$C^B(P, \gamma P) = \min_{0 \leq \alpha, \rho \leq 1} \left\{ C \left( \frac{a^2(1 - \alpha)P}{N} \right) + C \left( \frac{\alpha(1 - \rho^2)P}{N} \right), C \left( \frac{(\alpha + b^2\gamma + 2b\rho\sqrt{\alpha\gamma})P}{N} \right) \right\}$$

$$\mathcal{E}_b^B = 2N \ln 2 \times \min \left\{ \frac{1 + a^2 + b^2}{a^2(1 + b^2)}, 1 \right\}$$

Achieved using following block Markov coding scheme:

- $X_R \sim \mathcal{N}(0, (1 - \alpha)P)$ and $X_1 \sim \mathcal{N}(0, \gamma P)$ are independent
- $X_D = \sqrt{\alpha} \left( \frac{\rho}{\sqrt{\gamma}} X_1 + X_D' \right)$, where $X_D' \sim \mathcal{N}(0, (1 - \rho^2)P)$ is independent of $X_R$ and $X_1$
- Send new information to relay through $X_R$, to receiver through $X_D'$
- $X_1$ and rest of $X_D$ send information coherently to remove the uncertainty of the receiver about the previous message
Conclusion

• For AWGN relay channel:
  – Found upper and lower bounds on $E_b$ whose ratio is $< 2$
  – Upper bound can be improved for $a \approx 1$ using quantization scheme
  – Finding $E_b$ does not seem to be easier than finding $C(P, \gamma P)$
  – We are exploring other coding schemes for better bounds

• When relay and sender use different frequency bands (Model A):
  – Bounds are tight when $b \leq 1$
  – Ratio of bounds $< 2$
  – Upper bound can be improved using quantization scheme

• When sender uses different frequency bands for relay and receiver (Model B):
  – Capacity and the minimum energy-per-bit are found