It turns out that the Ising model (on the infinite lattice) is well-defined for some range of its local characteristics and that it can equivalently be described as having the limiting distribution as $M, N \to \infty$ of the following field defined on an orthogonal $M \times N$ subset of the infinite lattice. (For more detailed (and also introductory) descriptions of the Ising model, see [5].)

The sample space Ω has 2^{MN} possible configurations ω each with an assigned probability $P(\omega)$. The distribution assumed is Gibbsian, i.e.,

$$P(\omega) = e^{-U(\omega)}/Z, \tag{1a}$$

where

$$U(\omega) = bn_0(\omega) + hn_1(\omega). \tag{1b}$$

The Gibbsian distribution could be a fruitful area of research in the fields of information theory or image processing [11], [12]. In our case the quantity $U(\omega)$, called the "energy" of the configuration, is defined in terms of the number $n_0(\omega)$ of odd bonds (i.e., the number of pairs of neighboring sites with nonequal values) and the total number $n_1(\omega)$ of ones appearing in the configuration ω. Finally, Z is a normalizing constant so that $\Sigma_\omega P(\omega) = 1$. In the terminology of statistical mechanics, Z is the partition function of the random field; it is a function of the coefficients b and h of (1). It turns out that all the statistical properties of the field are expressible in terms of the partition function Z. It also turns out that the Gibbsian measure provides the field with the property of maximum entropy among all fields having the same expected value of the energy. This entropy is given (in terms of the partition function) by

$$H(P) = \log Z + E\{U\} = \log Z - \frac{1}{Z}\left(b\frac{\partial Z}{\partial b} + h\frac{\partial Z}{\partial h}\right). \tag{2}$$

Equation (2) remains valid in the limiting distribution of the infinite stationary random field (i.e., as $M, N \to \infty$ where each of the terms in (2) is considered divided by MN), which is well-defined for some range of values of b and h. This limit gives the entropy *rate* of the field. The calculation of the partition function Z of the limiting distribution for $h = 0$ was first solved in 1944 by Onsager [6], who succeeded in this evaluation using a very involved algebraic approach. Today several much simpler approaches to the problem exist [7]. We mention the result for the entropy rate that we found using (2) and the partition function Z as reported by Feynman [8, eq. (5.29 a)], in the case that $h = 0$; we have done some manipulations on the coefficients in order to match our notation:

$$\log_2 Z = 1 + \frac{1}{2}\log_2 y + \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \log_2\left[y + \frac{1}{y} - (\cos\xi + \cos\eta)\right]\frac{d\xi\, d\eta}{(2\pi)^2}, \tag{3}$$

in which $y = \sinh(b)$. From (2) and (3) we find the following formula for the entropy rate measured in bits per pel:

$$H(b) = 1 + \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\log_2\left[y^2 + 1 - y(\cos\xi + \cos\eta)\right]\frac{d\xi\, d\eta}{(2\pi)^2} - \frac{1}{2}\,\frac{b\cosh(b)}{\ln 2}\int_0^{2\pi}\int_0^{2\pi}\frac{2y - (\cos\xi + \cos\eta)}{y^2 + (\cos\xi + \cos\eta)}\,\frac{d\xi\, d\eta}{(2\pi)^2}.$$

This means that if a black-and-white image has the statistical characteristics of the Ising model, then its maximum compression in exact form, measured in bits/pixel, would never be better than the one given by the equation above, however sophisticated a coding scheme one uses.

It should again be noted that the entropy representation of the previous section is equal to $H(b)$ as shown above. In [4] $H(b)$ is evaluated for various values of b, and an encoding scheme is proposed whose compression performance for encoding sample fields of the Ising model is near optimum, i.e., is very close to the one indicated by the entropy rate.

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The Capacity Region of a Class of Deterministic Interference Channels

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MAX HENRIQUE M. COSTA

Abstract—The capacity region of a class of deterministic discrete memoryless interference channels is established. In this class of channels the outputs Y_1 and Y_2 are (deterministic) functions of the inputs X_1 and X_2 such that $H(Y_1 \mid X_1) = H(V_2)$ and $H(Y_2 \mid X_2) = H(V_1)$ for all product probability distributions on X_1X_2, where V_1 is a function of X_1 and V_2 a function of X_2. The capacity region for the case in which $V_2 = 0$ and Y_1 depends randomly on X_1 is also obtained and illustrated with an example.

I. Introduction

The interference channel (IFC) models the communication between M transmitter–receiver pairs in which the ith transmitter wishes to send information reliably to the ith receiver in the presence of interference from the other senders. It was first considered by Shannon [1] and has been investigated by Ahlswede [2], Carleial [3]–[5], Sato [6], [8], Han and Kobayashi [7], among other researchers. In this correspondence we shall restrict our attention to the $M = 2$ case.

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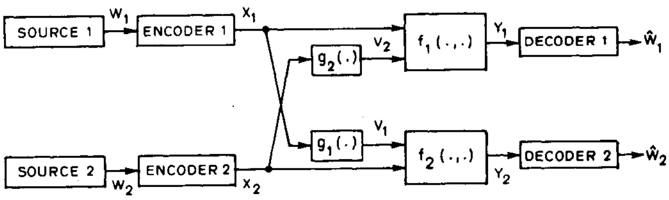

Fig. 1. Class of IFC's under investigation.

The capacity region for the general IFC is still unknown. It has been obtained for the following special cases:

1) when all the outputs are statistically equivalent [2], [3], [6];
2) for the strong interference case of the Gaussian IFC [7], [8];
3) for a class of discrete additive degraded IFC's [9].

In case 1) the capacity region coincides with that of the multiple access channel (MAC) with senders X_1, X_2 and receiver Y_1 (or Y_2). The capacity region for case 2) is also related to those of the underlying MAC's. This result can be extended to discrete memoryless IFC's for which $I(X_1; Y_2) \geq I(X_1; Y_1)$ and $I(X_2; Y_1) \geq I(X_2; Y_2)$ for all product probability distributions (PD's) on X_1X_2. For case 3) the capacity region is equal to that of the degraded broadcast channel (DBC) with input $X_1 + X_2$ and outputs Y_1 and Y_2.

In this correspondence, we establish the capacity region of the class of deterministic discrete memoryless IFC's depicted in Fig. 1, in which the outputs Y_1 and Y_2 and the interferences V_1 and V_2 are (deterministic) functions of the inputs X_1 and X_2:

$$Y_1 = f_1(X_1, V_2) \tag{1}$$

$$Y_2 = f_2(X_2, V_1) \tag{2}$$

$$V_1 = g_1(X_1) \tag{3}$$

$$V_2 = g_2(X_2), \tag{4}$$

where $f_1(\cdot,\cdot)$ and $f_2(\cdot,\cdot)$ satisfy the conditions

$$H(Y_1 \mid X_1) = H(V_2) \tag{5}$$

and

$$H(Y_2 \mid X_2) = H(V_1) \tag{6}$$

for all product PD's on X_1X_2. These conditions are equivalent to requiring the existence of functions $h_1(\cdot,\cdot)$ and $h_2(\cdot,\cdot)$ such that $V_2 = h_1(X_1, Y_1)$ and $V_1 = h_2(X_2, Y_2)$.

As an extension of this result, we obtain the capacity region for the case in which $V_2 \equiv 0$ and Y_1 depends randomly on X_1 and illustrate it with an example.

II. Preliminaries

Consider the IFC shown in Fig. 1. There are two independent and uniformly distributed sources, one at sender 1 producing an integer $W_1 \in \boldsymbol{M}_1 = \{1,\cdots,M_1\}$ and the other at sender 2 producing an integer $W_2 \in \boldsymbol{M}_2 = \{1,\cdots,M_2\}$. Encoder 1 maps W_1 into $\boldsymbol{X}_1 = (X_{11},\cdots,X_{1n})$ and encoder 2 maps W_2 into $\boldsymbol{X}_2 = (X_{21},\cdots,X_{2n})$. The channel itself consists of four finite alphabets $\boldsymbol{X}_1 = \{1,\cdots,I\}$, $\boldsymbol{X}_2 = \{1,\cdots,J\}$, $\boldsymbol{Y}_1 = \{1,\cdots,K\}$, $\boldsymbol{Y}_2 = \{1,\cdots,L\}$ and four deterministic functions in agreement with (1)–(6).

Remark: We note that if $g_1(\cdot)$ is one-to-one, it will also be onto and therefore invertible. Then X_1 and V_1 will have identical entropies and we could, without loss of generality, disregard $g_1(\cdot)$ or assume it to be of the identity function. Only when $g_1(\cdot)$ is not invertible is it of interest to us. The same, of course, is valid for $g_2(\cdot)$.

Let an (M_1, M_2, n, λ_n) code for this channel be a set of two encoding functions e_1: $\boldsymbol{M}_1 \to \boldsymbol{X}_1^n$, e_2: $\boldsymbol{M}_2 \to \boldsymbol{X}_2^n$, and two decoding functions d_1: $\boldsymbol{Y}_1^n \to \boldsymbol{M}_1$, d_2: $\boldsymbol{Y}_2^n \to \boldsymbol{M}_2$ such that

$$p_{e,1}^n = \sum_{w_1, w_2} P\{d_1(\boldsymbol{Y}_1) \neq w_1 \mid W_1 = w_1, W_2 = w_2\}$$

$$p_{e,2}^n = \sum_{w_1, w_2} P\{d_2(\boldsymbol{Y}_2) \neq w_2 \mid W_1 = w_1, W_2 = w_2\}$$

and

$$\max\{p_{e,1}^n, p_{e,2}^n\} \equiv \lambda_n.$$

A rate pair (R_1, R_2) is said to be *achievable* if there is a sequence of $(2^{nR_1}, 2^{nR_2}, n, \lambda_n)$ codes with $\lambda_n \to 0$ as $n \to \infty$. The *capacity region* of this channel is defined as the closure of the set of all achievable rate pairs.

III. Determination of the Capacity Region

Theorem 1: Let C denote the capacity region of the channel of Fig. 1 satisfying (5) and (6). C is equal to the union of the set of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq H(Y_1 \mid V_2) \tag{7}$$

$$R_2 \leq H(Y_2 \mid V_1) \tag{8}$$

$$R_1 + R_2 \leq H(Y_1 \mid V_1V_2) + H(Y_2) \tag{9.a}$$

$$R_1 + R_2 \leq H(Y_1) + H(Y_2 \mid V_1V_2) \tag{9.b}$$

$$R_1 + R_2 \leq H(Y_1 \mid V_1) + H(Y_2 \mid V_2) \tag{9.c}$$

$$2R_1 + R_2 \leq H(Y_1) + H(Y_1 \mid V_1V_2) + H(Y_2 \mid V_2) \tag{10}$$

$$R_1 + 2R_2 \leq H(Y_1 \mid V_1) + H(Y_2) + H(Y_2 \mid V_1V_2) \tag{11}$$

over all product PD's on X_1X_2.

Proof

1) *Achievability*: This part follows easily from the inclusion of C in the region of Han and Kobayashi [7].

2) *Converse*: First we note that, from the convexity of the entropy function, it follows that C is a convex region.

Constraints (7) and (8) are identical to the ones in the usual outer bound for the general IFC [6], thus requiring no proof. We proceed to prove constraints (9)–(11). From Fano's inequality, we have

$$H(W_1 \mid \boldsymbol{Y}_1) \leq nR_1p_{e,1}^n + h(p_{e,1}^n) \equiv n\epsilon_{1n} \tag{12}$$

$$H(W_2 \mid \boldsymbol{Y}_2) \leq nR_2p_{e,2}^n + h(p_{e,2}^n) \equiv n\epsilon_{2n}, \tag{13}$$

where $h(\cdot)$ is the binary entropy function and $\epsilon_{1n}, \epsilon_{2n} \to 0$ as $n \to \infty$. Now consider

$$\begin{aligned} n(R_1 + R_2) &= H(W_1) + H(W_2) \\ &= I(W_1; \boldsymbol{Y}_1) + I(W_2; \boldsymbol{Y}_2) + H(W_1 \mid \boldsymbol{Y}_1) \\ &\quad + H(W_2 \mid \boldsymbol{Y}_2). \end{aligned}$$

Substituting from (19) and (20) we get

$$\begin{aligned} &n(R_1 + R_2) \\ &\leq I(W_1; \boldsymbol{Y}_1) + I(W_2; \boldsymbol{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq I(\boldsymbol{X}_1; \boldsymbol{Y}_1) + I(\boldsymbol{X}_2; \boldsymbol{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq I(\boldsymbol{X}_1; \boldsymbol{Y}_1 \mid \boldsymbol{V}_2) + I(\boldsymbol{X}_2; \boldsymbol{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq I(\boldsymbol{X}_1; \boldsymbol{V}_1\boldsymbol{Y}_1 \mid \boldsymbol{V}_2) + H(\boldsymbol{Y}_2) - H(\boldsymbol{Y}_2 \mid \boldsymbol{X}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq I(\boldsymbol{X}_1; \boldsymbol{V}_1 \mid \boldsymbol{V}_2) + I(\boldsymbol{X}_1; \boldsymbol{Y}_1 \mid \boldsymbol{V}_1\boldsymbol{V}_2) \\ &\quad + H(\boldsymbol{Y}_2) - H(\boldsymbol{V}_1) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &= H(\boldsymbol{Y}_1 \mid \boldsymbol{V}_1\boldsymbol{V}_2) + H(\boldsymbol{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq \sum_{i=1}^{n} [H(Y_{1i} \mid V_{1i}V_{2i}) + H(Y_{2i}) + \epsilon_{1n} + \epsilon_{2n}]. \end{aligned} \tag{14}$$

In the development above we made use of assumptions (5), (6), and the independence between X_1V_1 and X_2V_2.

In a completely analogous way, we obtain

$$n(R_1 + R_2) \leq \sum_{i=1}^{n} \left[H(Y_{1i}) + H(Y_{2i} \mid V_{1i}V_{2i}) + \epsilon_{1n} + \epsilon_{2n} \right]. \tag{15}$$

Using (14), we get

$$\begin{aligned} n(R_1 + R_2) &\leq I(\mathbf{X}_1; \mathbf{Y}_1) + I(\mathbf{X}_2; \mathbf{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &= H(\mathbf{Y}_1) - H(\mathbf{V}_1) + H(\mathbf{Y}_2) - H(\mathbf{V}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq H(\mathbf{Y}_1) - I(\mathbf{V}_1; \mathbf{Y}_1) + H(\mathbf{Y}_2) - I(\mathbf{V}_2; \mathbf{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &= H(\mathbf{Y}_1 \mid \mathbf{V}_1) + H(\mathbf{Y}_2 \mid \mathbf{V}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq \sum_{i=1}^{n} \left[H(Y_{1i} \mid V_{1i}) + H(Y_{2i} \mid V_{2i}) + \epsilon_{1n} + \epsilon_{2n} \right]. \end{aligned} \tag{16}$$

Next consider the inequality

$$n(2R_1 + R_2) \leq 2H(W_1) + H(W_2).$$

Substituting from (12) and (13), we obtain

$$\begin{aligned} n(2R_1 + R_2) &\leq 2I(W_1; \mathbf{Y}_1) + I(W_2; \mathbf{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq I(\mathbf{X}_1; \mathbf{Y}_1) + I(\mathbf{X}_1; \mathbf{Y}_1\mathbf{V}_1 \mid \mathbf{V}_2) + I(\mathbf{X}_2; \mathbf{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq H(\mathbf{Y}_1) - H(\mathbf{V}_1) + H(\mathbf{V}_1) + H(\mathbf{Y}_1 \mid \mathbf{V}_1\mathbf{V}_2) \\ &\quad + H(\mathbf{Y}_2) - H(\mathbf{V}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq H(\mathbf{Y}_1) + H(\mathbf{Y}_1 \mid \mathbf{V}_1\mathbf{V}_2) + H(\mathbf{Y}_2 \mid \mathbf{V}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq \sum_{i=1}^{n} \left[H(Y_{1i}) + H(Y_{1i} \mid V_{1i}V_{2i}) + H(Y_{2i} \mid V_{2i}) + 2\epsilon_{1n} + \epsilon_{2n} \right] \end{aligned} \tag{17}$$

and analogously

$$n(R_1 + 2R_2) \leq \sum_{i=1}^{n} \left[H(Y_{2i} \mid V_{2i}) + H(Y_{2i}) + H(Y_{2i} \mid V_{1i}V_{2i}) + \epsilon_{1n} + 2\epsilon_{2n} \right]. \tag{18}$$

Now, letting $n \to \infty$, we combine (14)–(18) and the convexity of C to conclude that a product PD on X_1X_2 exists for which (7)–(11) hold. This completes the proof of Theorem 1.

Remark: It is interesting to note that the simultaneous superposition coding described by Han and Kobayashi [7] is not needed to achieve the rates of Theorem 1. Since each encoder uses no more than one auxiliary random variable (V_1 for encoder 1 and V_2 for encoder 2), only sequential superposition, as used by Carleial, is needed to attain these rates. Nevertheless, proving a converse for the achievable region expressed by Carleial is extremely difficult. This is a good example of how a modification in the way a region of rates is expressed can affect the difficulty involved in establishing a converse.

IV. Extension

If we make $V_2 \equiv 0$ and allow Y_1 to be random according to $p(y_1 \mid x_1)$, we get the IFC shown in Fig. 2. If condition (6) still holds for $f_2(\cdot, \cdot)$ we can extend the result of Theorem 1 and obtain the capacity of this channel.

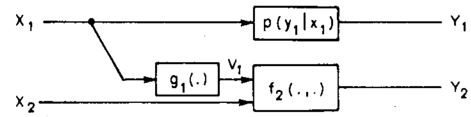

Fig. 2. Variation of IFC in Fig. 1.

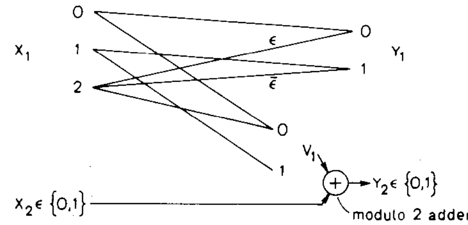

Fig. 3. Example of IFC in Fig. 2.

Theorem 2: The capacity region of the channel of Fig. 2 satisfying (6) is given by the union of all pairs (R_1, R_2) for which

$$R_1 \leq I(X_1; Y_1) \tag{19}$$

$$R_2 \leq H(Y_2 \mid V_1) \tag{20}$$

$$R_1 + R_2 \leq I(X_1; Y_1 \mid V_1) + H(Y_2)$$

over all product PD's on X_1X_2.

Proof

1) *Achievability*: This follows immediately from the inclusion in the Han and Kobayashi region [7].

2) *Converse*: Constraints (19) and (20) follow immediately. Application of Fano's inequality and the data processing inequality yields

$$\begin{aligned} n(R_1 + R_2) &= H(W_1) + H(W_2) \\ &\leq I(\mathbf{X}_1; \mathbf{Y}_1) + (\mathbf{X}_2; \mathbf{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq I(\mathbf{X}_1; \mathbf{V}_1\mathbf{Y}_1) + H(\mathbf{Y}_2) - H(\mathbf{Y}_2 \mid \mathbf{X}_2) \\ &\quad + n(\epsilon_{1n} + \epsilon_{2n}) \\ &= I(\mathbf{X}_1; \mathbf{Y}_1 \mid \mathbf{V}_1) + H(\mathbf{Y}_2) + n(\epsilon_{1n} + \epsilon_{2n}) \\ &\leq \sum_{i=1}^{n} \left[I(X_{1i}; Y_{1i} \mid V_{1i}) + H(Y_{2i}) + \epsilon_{1n} + \epsilon_{2n} \right]. \end{aligned} \tag{21}$$

It is easy to show the convexity of the region in this theorem. From this convexity and (21) the converse follows.

Example: Fig. 3 shows an example of this one-sided IFC. Note that if $\epsilon = 0$, perfect communication (i.e., 1 bit per channel use) can be achieved by both sender–receiver pairs if sender 1 restricts his alphabet to the set $\{0, 2\}$. On the other hand, if $\epsilon = 1$ the effect of interference becomes devastating and the capacity region is the triangle achieved by time-sharing between rate pairs (0, 1) and (1, 0). The capacity region of this channel when $\epsilon = 0.1$ is shown in Fig. 4.

V. Discussion

We can interpret conditions (5) and (6) as a requirement that each receiver, after having decoded his sender's message, will be able to know exactly the interference caused by the other sender. This observation follows immediately by noting that conditions (5) and (6) are equivalent to requiring the functions $f_1(x_1, \cdot)$ in (1) and $f_2(x_2, \cdot)$ in (2) to be one-to-one mappings for each $x_1 \in \mathbf{X}_1$ and each $x_2 \in \mathbf{X}_2$.

In general, a receiver will not be able to decode the message addressed to the other receiver. In this respect, the class of channels studied in this correspondence differs from all the IFC's

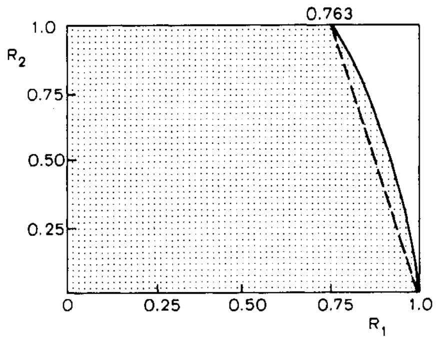

Fig. 4. Capacity region of IFC in Fig. 3 with $\epsilon = 0.1$.

for which capacity regions have been established. For those channels, at least one of the receivers is sure to decode the message interfering in his communication.

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Multigram Codes

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***Abstract*—A multigram code is a list of codewords for multigrams (of various lengths) belonging to a set S. Three interrelated problems require designing a good set S, good codewords, and a good strategy for dissecting messages into multigrams of S. Focus is placed on the dissection strategy and applications to English.**

I. Introduction

An n-gram block code for a source alphabet of A letters contains codewords for all A^n n-grams, a large number unless n is small. A shorter code might provide codewords only for important n-grams and extra codewords for enough single letters and short blocks so that every message can be encoded. Thus, with *multigram* used to mean a block of any length, a *multigram code* is a list of codewords for multigrams belonging to some set S, with members not necessarily all of the same length.

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Lists of abbreviations, used for telegraph text compression or secrecy in the 19th century and for semaphore signaling in the 18th, were essentially multigram codes (Phillips [1], Shay [2], Slater [3], Friedman [4]). The dictionary codes of Schwartz [5] and White [6] are modern computer-oriented examples. White estimated a 50 percent compression of English text using a set S of 1000 multigrams (a much better compression than could be achieved even by coding all 19 683 trigrams). Some approaches to complexity theory and universal coding use multigram codes (Lempel and Ziv [7], [8]).

The encoder must dissect the message into a sequence of multigrams from S, which are then replaced by codewords. Although S can be designed so that only one dissection into multigrams is possible (for example the variable-length-to-block codes of Jelinek and Schneider [9]), that procedure constrains S severely. A more typical set S might contain TH, HE, and all single letters; then THE can be dissected as TH/E, T/HE, or T/H/E (slashes indicate dissection cuts). The part of the encoder that makes the dissection (called the *dissector* here) requires a rule, or *dissection strategy*, to choose one particular dissection.

Normally, one would encode multigrams of S into digits, or other channel symbols, by a uniquely decipherable code such as the rate-maximizing code of Huffman [10]. However, unique decipherability is not strictly required. It allows the receiver to decipher both the message and the dissection. The dissection then represents unwanted transmitted information that can sometimes be avoided. For example, if the dissector always leaves the diagram TH intact, then the code sequence for T/H is an *alias* that is never transmitted; it can be used without ambiguity as the codeword for another multigram.

The elements of an encoder—S, the dissector, and the codewords—are highly interrelated. To optimize all three together seems difficult. This correspondence gives partial results, mostly about dissectors, and an application to simple English text codes.

II. Dissectors

Simple dissection strategies come to mind immediately. A *block dissector* first cuts the message into blocks of some constant manageable size k and then further subdivides each block into elements of S in a way that uses fewest digits. A *greedy dissector* cuts consecutive multigrams from the message one at a time, always postponing the next cut to make the next multigram as long as possible. With $S = \{E, H, O, R, T, OT, THER\}$ the greedy dissector dissects OTHER as OT/H/E/R. A *priority dissector* keeps the members of S in a priority ordering $\alpha_1 > \alpha_2 > \cdots > \alpha_r$, where $\alpha > \beta$ means that cutting out α intact is preferable to cutting out β intact. The dissector scans the entire message, cutting out α_1 where possible. Then it scans the remaining pieces of message, cutting out α_2, etc. If THER > OT the priority dissector finds the dissection O/THER that the greedy dissector misses.

Unlike greedy dissectors, priority dissectors work "off-line;" they cannot act until the message ends. That entails a long coding delay and requires memory to store the entire message. To encode off-line one could, with little extra complication, use an *optimal dissector* that encodes the message into the fewest possible digits, or other channel symbols. Here the codewords for multigrams $\alpha \in S$ are assumed given.

Theorem 1: The number of operations, which is required to dissect a message optimally p letters long, grows at most linearly with p.

Proof: Let $n(\alpha)$ denote the number of letters in a multigram $\alpha \in S$ and $m(\alpha)$ the number of digits in the codeword for α.

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