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## A Proof of Marton's Coding Theorem for the Discrete Memoryless Broadcast Channel

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***Abstract*—A simple proof using random partitions and typicality is given for Marton's coding theorem for broadcast channels.**

In [1] Marton established an achievable-rate region for the discrete memoryless broadcast channel (DMBC). Her proof uses a random coding method which is a combination of the coding techniques of Bergmans [2], Cover [3], and van der Meulen [4], together with the random coding technique used to prove source coding theorems in rate-distortion theory. Her theorem is a generalization of the results of Cover [3] and van der Meulen [4] on this problem and the best inner bound to the capacity region known to date. Moreover, as is proved in [1], her result is tight for broadcast channels having one deterministic component.

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Independently and at about the same time, Pinsker [5] established the capacity region for noiseless DMBC's with $l \geqslant 2$ outputs. Specializing to the case of a DMBC with two deterministic components, the regions of Marton [1] and Pinsker [5] coincide. Pinsker's proof is based on a random partitioning of the output spaces together with a combinatorial argument implicitly involving typical output sequences. His proof, however, is tied to the deterministic aspects of this particular DMBC and does not seem to carry over to arbitrary DMBC's. Pinsker's result generalizes an earlier result by Gelfand [6] who established the capacity region of the so-called Blackwell channel (see also [4] and [7]). Gelfand's proof is based on the method of defects as explored by Kuznetsov and Tsybakov [8].

In this correspondence, we establish a new proof of the Marton region which we believe to be more transparent than the one given in [1]. Our proof involves standard random coding, the technique of random partitions as developed in [9], [10], and a new simple joint-typicality lemma.

We begin with some definitions. We consider a DMBC $(\mathcal{X}, p(y,z|x), \mathcal{Y}\times\mathcal{Z})$ and use the now standard definition of a code with average probability of error ([1], [3], [4], [7]). $(R_1, R_2)$ denotes the pair of achievable rates also defined in the usual way. The capacity region of the DMBC is the set of all achievable rates. The random variables $U$ and $V$ are auxiliary random variables with finite ranges defined in the sense of [1]. We assume that the reader is familiar with the joint typicality approach used in [3], [9], and [10]. Our goal is to prove the following theorem which we believe to be the essence of Theorem 2 of [1].

*Theorem (Marton [1]):* Let

$$\begin{aligned}\mathcal{R}_0 = \{(R_1, R_2) &: R_1, R_2 \geqslant 0,\\ R_1 &\leqslant I(U;Y),\\ R_2 &\leqslant I(V;Z),\\ R_1 + R_2 &\leqslant I(U;Y) + I(V;Z) - I(U;V),\\ &\{\text{for some } p(u,v,x) \text{ on } \mathcal{U}\times\mathcal{V}\times\mathcal{X}\}. \end{aligned} \tag{1}$$

Then any rate pair $(R_1, R_2) \in R_0$ is achievable for the DMBC $(\mathcal{X}, p(y,z|x), \mathcal{Y}\times\mathcal{Z})$.

*Proof:* Let $\epsilon > 0$ and $n \geqslant 1$ be given. Let $A_\epsilon(U)$ denote the set of $\epsilon$-typical $n$-sequences $\boldsymbol{u} \in \mathcal{U}^n$. Generate $2^{n(I(U;Y)-\epsilon)}$ independent identically distributed (i.i.d.) sequences $\boldsymbol{u}$ each with probability

$$P(\boldsymbol{u}) = \begin{cases} \dfrac{1}{\|A_\epsilon(U)\|}, & \boldsymbol{u} \in A_\epsilon(U),\\ 0, & \text{otherwise,}\end{cases} \tag{2}$$

where $\|A\|$ denotes the cardinality of the set $A$. Label these $\boldsymbol{u}(k)$, $k \in [1, 2^{n(I(U;Y)-\epsilon)}]$. Also, generate $2^{n(I(V;Z)-\epsilon)}$ i.i.d. sequences $\boldsymbol{v}$ each with probability

$$P(\boldsymbol{v}) = \begin{cases} \dfrac{1}{\|A_\epsilon(V)\|}, & \boldsymbol{v} \in A_\epsilon(V),\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

Label these $\boldsymbol{v}(l)$, $l \in [1, 2^{n(I(V;Z)-\epsilon)}]$.

Next, for $i \in [1, 2^{nR_1}]$, define the cells

$$B_i = \left[(i-1)\cdot 2^{n(I(U;Y)-R_1-\epsilon)} + 1,\ i \cdot 2^{n(I(U;Y)-R_1-\epsilon)}\right].$$

Similarly, for $j \in [1, 2^{nR_2}]$, define the cells

$$C_j = \left[(j-1)\cdot 2^{n(I(V;Z)-R_2-\epsilon)} + 1,\ j \cdot 2^{n(I(V;Z)-R_2-\epsilon)}\right],$$

where without loss of generality $2^{n(I(U;Y)-R_1-\epsilon)}$ and $2^{n(I(V;Z)-R_2-\epsilon)}$ are considered to be integer valued.

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Define for every $(i,j)\in[1,2^{nR_1}]\times[1,2^{nR_2}]$ the set

$$\mathcal{Y}_{ij}=\{(U(k),V(l)): k\in B_i, l\in C_j, (U(k),V(l))\in A_\epsilon\} \quad (4)$$

consisting of the jointly $\epsilon$-typical $(\boldsymbol{u}(k),\boldsymbol{v}(l))$ sequences in $B_i\times C_j$.

We now bound the probability that the Cartesian product $B_i\times C_j$ does not contain a pair $(\boldsymbol{u}(k),\boldsymbol{v}(l))$ which is jointly $\epsilon$-typical, i.e., the probability that $\mathcal{Y}_{ij}$ is empty.

*Lemma:* For any particular cell $B_i$, any particular cell $C_j$, $\epsilon>0$, and sufficiently large $n$:

$$P\{\|\mathcal{Y}_{ij}\|=0\}\leqslant\epsilon/4, \quad (5)$$

provided

$$R_1+R_2<I(U;Y)+I(V;Z)-I(U;V)-2\epsilon-\delta(\epsilon), \quad (6)$$

where $\delta(\epsilon)\to 0$ as $\epsilon\to 0$.

*Proof:* Using Chebychev's inequality, it is easy to see that

$$P\{\|\mathcal{Y}_{11}\|=0\}\leqslant P\{|\|\mathcal{Y}_{11}\|-E\|\mathcal{Y}_{11}\||>\epsilon E\|\mathcal{Y}_{11}\|\}\leqslant\frac{\operatorname{var}(\|\mathcal{Y}_{11}\|)}{\epsilon^2(E\|\mathcal{Y}_{11}\|)^2}. \quad (7)$$

Now, to obtain bounds on var $(\|\mathcal{Y}_{11}\|)$ and $E(\|\mathcal{Y}_{11}\|)$, define the indicator functions

$$1((U(k),V(l))\in\mathcal{Y}_{11})=\begin{cases}1, & (U(k),V(l))\in\mathcal{Y}_{11},\\ 0, & \text{otherwise.}\end{cases}$$

The cardinality of the set $\mathcal{Y}_{11}$ is given by

$$\|\mathcal{Y}_{11}\|=\sum_{\substack{k\in B_1\\ l\in C_1}}1((U(k),V(l))\in\mathcal{Y}_{11}).$$

But since $E\{1((U(k),V(l))\in\mathcal{Y}_{11})\}\geqslant 2^{-n(I(U;V)+\delta(\epsilon))}$ where $\delta(\epsilon)\to 0$ as $\epsilon\to 0$, it follows that

$$E(\|\mathcal{Y}_{11}\|)\geqslant\|B_1\|\cdot\|C_1\|2^{-n(I(U;V)+\delta(\epsilon))}\geqslant 2^{n(I(U;Y)+I(V;Z)-R_1-R_2-I(U;V)-2\epsilon-\delta(\epsilon))}.$$

Evaluating the variance (see Appendix) we obtain that

$$\operatorname{var}(\|\mathcal{Y}_{11}\|)\leqslant 2^{n(I(U;Y)+I(V;Z)-R_1-R_2-I(U;V)-2\epsilon+\delta(\epsilon))}.$$

Therefore for sufficiently large $n$

$$P\{\|\mathcal{Y}_{11}\|=0\}\leqslant\epsilon/4,$$

provided

$$R_1+R_2<I(U;Y)+I(V;Z)-I(U;V)-2\epsilon-\delta(\epsilon),$$

and the Lemma is proved.

Now consider the following encoding–decoding method.

*Encoding:* If a message pair $(i,j)$ is to be transmitted, pick one pair $(\boldsymbol{u}(k),\boldsymbol{v}(l))\in A_\epsilon(U,V)\cap(B_i\times C_j)$. Then find an $\boldsymbol{x}(i,j)$ which is jointly $\epsilon$-typical with that pair $(\boldsymbol{u}(k),\boldsymbol{v}(l))$ and designate it as the codeword corresponding to $(i,j)$.

*Decoding:* Receiver $y$ upon receiving $\boldsymbol{y}$ finds the unique index $k$ such that $\boldsymbol{u}(k)$ is jointly $\epsilon$-typical with $\boldsymbol{y}$. Similarly, receiver $z$ upon receiving $\boldsymbol{z}$, finds the unique index $l$ such that $\boldsymbol{v}(l)$ is jointly $\epsilon$-typical with $\boldsymbol{z}$.

### *Calculation of Probability of Error*

An error will be declared if one or more of the following events occur.

- $E_1$ The encoding step fails; there does not exist a pair $(\boldsymbol{u}(k),\boldsymbol{v}(l))\in(B_i\times C_j)\cap A_\epsilon(U,V)$.
- $E_2$ $(\boldsymbol{u}(k),\boldsymbol{v}(l),\boldsymbol{x}(i,j),\boldsymbol{y},\boldsymbol{z})\notin A_\epsilon(U,V,X,Y,Z)$.
- $E_3$ The decoding step 1 fails, there exists $\hat{k}\neq k$ such that $(\boldsymbol{u}(\hat{k}),\boldsymbol{y})\in A_\epsilon(U,Y)$.
- $E_4$ The decoding step 2 fails, there exists $\hat{l}\neq l$ such that $(\boldsymbol{v}(\hat{l}),\boldsymbol{z})\in A_\epsilon(V,Z)$.

It is now easy to see that, for $n$ sufficiently large

i) $P(E_1)\leqslant\epsilon/4$, if $R_1+R_2<I(U;Y)+I(V;Z)-I(U;V)-2\epsilon-\delta$;
ii) $P(E_2)\leqslant\epsilon/4$;
iii) $P(E_3)\leqslant\epsilon/4$, if $R_1<I(U;Y)-\epsilon$;
iv) $P(E_4)\leqslant\epsilon/4$, if $R_2<I(V;Z)-\epsilon$.

Therefore

$$\text{error probability}=P\left(\bigcup_{i=1}^{4}E_i\right)\leqslant\sum_{i=1}^{4}P(E_i)\leqslant\epsilon.$$

This proves the theorem.

*Remark 1:* Notice that our proof is symmetric in $U$ and $V$ and does not require time-sharing as in [1].

*Remark 2:* Our proof can be easily extended to prove Marton's Theorem 2.


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Helpful discussions with Professor Thomas M. Cover and Chris Heegard are gratefully acknowledged.


## Appendix

*Bound on var* $(\|\mathcal{Y}_{11}\|)$:

$$\begin{aligned}
\|\mathcal{Y}_{11}\|^2 &=\Bigg(\sum_{\substack{k\in B_1\\ l\in C_1}}1((U(k),V(l))\in\mathcal{Y}_{11})\Bigg)^2\\
&=\sum_{\substack{k_1\in B_1\\ l_1\in C_1}}\sum_{\substack{k_2\in B_1\\ l_2\in C_1}}1((U(k_1),V(l_1))\in\mathcal{Y}_{11},(U(k_2),V(l_2))\in\mathcal{Y}_{11})\\
&=\sum_{\substack{k_1=k_2\in B_1\\ l_1=l_2\in C_1}}1((U(k_1),V(l_1)\in\mathcal{Y}_{11})\\
&\quad+\sum_{\substack{k_1\neq k_2\in B_1\\ l_1=l_2\in C_1}}1((U(k_1),V(l_1))\in\mathcal{Y}_{11},(U(k_2),V(l_1))\in\mathcal{Y}_{11})\\
&\quad+\sum_{\substack{k_1=k_2\in B_1\\ l_1\neq l_2\in C_1}}1((U(k_1),V(l_1))\in\mathcal{Y}_{11},(U(k_1),V(l_2))\in\mathcal{Y}_{11})\\
&\quad+\sum_{\substack{k_1\neq k_2\in B_1\\ l_1\neq l_2\in C_1}}1((U(k_1),V(l_1))\in\mathcal{Y}_{11},(U(k_2),V(l_2))\in\mathcal{Y}_{11}).
\end{aligned} \quad (8)$$

Taking expectations, we obtain

$$\begin{aligned}
E(\|\mathcal{Y}_{11}\|^2)&=\|B_1\|\cdot\|C_1\|P\{(U(k),V(l))\in\mathcal{Y}_{11}\}\\
&\quad+\|C_1\|(\|B_1\|^2-\|B_1\|)\\
&\quad\cdot P\{(U(k_1),V(l))\in\mathcal{Y}_{11},(U(k_2),V(l))\in\mathcal{Y}_{11}\}\\
&\quad+\|B_1\|(\|C_1\|^2-\|C_1\|)\\
&\quad\cdot P\{(U(k),V(l_1))\in\mathcal{Y}_{11},(U(k),V(l_2))\in\mathcal{Y}_{11}\}\\
&\quad+(\|B_1\|^2-\|B_1\|)(\|C_1\|^2-\|C_1\|)\\
&\quad\cdot P\{(U(k_1),V(l_1)\in\mathcal{Y}_{11},(U(k_2),V(l_2)\in\mathcal{Y}_{11}\}.
\end{aligned} \quad (9)$$

It is easily seen that

i) $P\{(U(k),V(l))\in\mathcal{Y}_{11}\}\leqslant 2^{-n(I(U;V)-\delta(\epsilon))}$;
ii) For $k_1\neq k_2$, $l_1\neq l_2$,

$$P\{(U(k_1),V(l_1))\in\mathcal{Y}_{11},(U(k_2),V(l_2))\in\mathcal{Y}_{11}\}\leqslant 2^{-2n(I(U;V)-\delta(\epsilon))};$$

iii) For $k_1 \neq k_2$,

$$P\{(U(k_1), V(l)) \in \mathcal{Y}_{11}, (U(k_2), V(l)) \in \mathcal{Y}_{11}\}$$
$$= E\big(P\{(U(k_1), v(l)) \in \mathcal{Y}_{11}\} P\{(U(k_2), v(l)) \in \mathcal{Y}_{11}\}\big)$$
$$\leq 2^{-2n(I(U;V)-\delta(\epsilon))};$$

iv) For $l_1 \neq l_2$,

$$P\{(U(k), v(l_1)) \in \mathcal{Y}_{11}, (U(k), v(l_2)) \in \mathcal{Y}_{11}\} \leq 2^{-2n(I(U;V)-\delta(\epsilon))}.$$

Substituting from i–iv, it follows that

$$\operatorname{var}(\|\mathcal{Y}_{11}\|) \leq 2^{n(I(U;Y)+I(V;Z)-R_1-R_2-I(U;V)-2\epsilon+\delta(\epsilon))},$$

and the upper bound is proved.

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# The Capacity Region for the Deterministic Broadcast Channel with a Common Message

TE SUN HAN


***Abstract*—The deterministic broadcast channel with two output terminals is studied for the case of a common message. The capacity region for this channel is established by means of a random coding argument combining the standard channel coding technique and the standard source coding technique.**


## I. Introduction

The broadcast channel was first studied by Cover [1] using the new technique of superposition coding. Since then the problem of determining the capacity region has been investigated extensively, and complete solutions have been given for several special classes such as degraded broadcast channels (e.g., Bergmans [2], Gallager [3], Ahlswede and Körner [4]), broadcast channels with degraded message sets (Körner and Marton [5]), capability-degraded broadcast channels (El Gamal [6]), and broadcast channels with one deterministic component but *without* common messages (Marton [7], Gelfand and Pinsker [8]). (The deterministic channel without *common* message sets has been studied by Gelfand [9] and Pinsker [10]). The capacity region for the deterministic broadcast channel *with* a common message is determined here by combining the standard channel coding technique and the standard source coding technique for correlated sources.

It was pointed out by the referees that the present result can be derived from independent (and as yet unpublished) results of Gelfand and Pinsker [11] and Marton [12].


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## II. Definitions and the Results

Let $\boldsymbol{B} = (\mathcal{X}, \omega_1, \omega_2, \mathcal{Y}, \mathcal{Z})$ be a broadcast channel which is discrete memoryless and stationary where $\mathcal{X}$ is a finite input alphabet, $\mathcal{Y}$ and $\mathcal{Z}$ are two finite output alphabets, and $\omega_1, \omega_2$ are channel probabilities for $\mathcal{Y}$, $\mathcal{Z}$, respectively:

$$\omega_1(y|x), \qquad \text{for } y \in \mathcal{Y}, \qquad x \in \mathcal{X}, \tag{2.1}$$

$$\omega_2(z|x), \qquad \text{for } z \in \mathcal{Z}, \qquad x \in \mathcal{X}. \tag{2.2}$$

Without loss of generality we assume that the channel components are *independent*, i.e., given an input letter $x$, two output letters $y$ and $z$ are generated independently of each other (cf. Cover [13]). If the values $\omega_1$, $\omega_2$ are all one or zero, i.e., $\omega_1$, $\omega_2$ are reduced to deterministic functions $\omega_1$: $\mathcal{X} \to \mathcal{Y}$, $\omega_2$: $\mathcal{X} \to \mathcal{Z}$, then $\boldsymbol{B}$ is called *deterministic*. The channel $\boldsymbol{B}$ maps an input $n$-sequence $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathcal{X}^n$ to a pair of output $n$-sequences $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathcal{Y}^n$ (to receiver one) and $\boldsymbol{z} = (z_1, \cdots, z_n) \in \mathcal{Z}^n$ (to receiver two) with conditional probabilities $\omega_1$ and $\omega_2$.

Three sources $W_1$, $W_0$, $W_2$ (random variables) are defined on $\mathcal{W}_1$, $\mathcal{W}_0$, $\mathcal{W}_2$, respectively: $\mathcal{W}_i = \{1, 2, \cdots, M_i\}$ $(i = 1, 0, 2)$, where $W_1$ and $W_2$ are *private* messages and $W_0$ is a *common* message. We assume that $W_1$, $W_0$, $W_2$ are generated independently and equiprobably over $\mathcal{W}_1$, $\mathcal{W}_0$, $\mathcal{W}_2$, respectively.

A code is a collection of codewords and decoding sets

$$\{\boldsymbol{x}_{ikj}; \mathcal{B}_{ik}, \mathcal{C}_{jk} | i \in \mathcal{W}_1, j \in \mathcal{W}_2, k \in \mathcal{W}_0\},$$

such that i) $\boldsymbol{x}_{ikj} \in \mathcal{X}^n$ (codewords), ii) $\mathcal{B}_{ik} \subseteq \mathcal{Y}^n$, $\mathcal{C}_{jk} \subseteq \mathcal{Z}^n$ (decoding sets), and iii) $\mathcal{B}_{ik} \cap \mathcal{B}_{i'k'} = \phi$ for $ik \neq i'k'$; $\mathcal{C}_{jk} \cap \mathcal{C}_{j'k'} = \phi$ for $jk \neq j'k'$. The mapping $\phi$: $\mathcal{W}_1 \times \mathcal{W}_0 \times \mathcal{W}_2 \to \mathcal{X}^n$ $(\phi(ikj) = \boldsymbol{x}_{ikj})$ is the encoding function, and the mappings $\psi_1: \mathcal{Y}^n \to \mathcal{W}_1 \times \mathcal{W}_0$ $(\psi_1(y) = ik$ for $y \in \mathcal{B}_{ik})$, $\psi_2: \mathcal{Z}^n \to \mathcal{W}_2 \times \mathcal{W}_0$ $(\psi_2(z) = jk$ for $z \in \mathcal{C}_{jk})$ are the decoding functions for the receivers one and two. The *mean* probabilities of decoding error for receivers one and two are indicated by $P_{e1}$, $P_{e2}$.

A triple $(R_1, R_0, R_2)$ of nonnegative real numbers is called *achievable* if for every $\eta > 0$, $0 < \lambda < 1$ and for sufficiently large $n$, there exists a code for which

$$R_a - \eta \leq (1/n) \log M_a, \qquad a = 1, 0, 2, \tag{2.3}$$

$$P_{el} \leq \lambda, \qquad P_{e2} \leq \lambda. \tag{2.4}$$

The set of all achievable rates $(R_1, R_0, R_2)$ is called the *capacity region* of the channel $\boldsymbol{B}$.

Let $U$, $X$, $Y$, $Z$ be random variables defined on $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ for $\boldsymbol{B}$ where $\mathcal{U}$ is an arbitrary finite set. We write $XYZ \in \mathcal{P}$ and $UXYZ \in \mathcal{P}$ if $UXYZ$ forms a Markov chain $U \to X \to YZ$, such that $Y$ and $Z$ are distributed according to the channel probabilities $\omega_1$, $\omega_2$ given $X$. Let $\mathcal{R}$ be the closure of the set

$$\begin{aligned}\{(R_1, R_0, R_2)\colon & R_1 \geq 0, R_0 \geq 0, R_2 \geq 0,\\ & R_0 \leq \min(I(U;Y), I(U;Z)),\\ & R_0 + R_1 \leq I(X;Y|U) + \min(I(U;Y), I(U;Z)),\\ & R_0 + R_2 \leq I(X;Z|U) + \min(I(U;Y), I(U;Z)),\\ & R_0 + R_1 + R_2 \leq I(X;YZ|U) + \min(I(U;Y), I(U;Z)),\\ & \text{for some } UXYZ \in \mathcal{P}\}.\end{aligned} \tag{2.5}$$