BROADCAST CHANNELS WITH AND WITHOUT FEEDBACK[†]

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Abstract

The capacity region for the discrete memoryless broadcast channel $p(y,z|x)$ for which $I(X;Y) \geq I(X;Z)$ for all input distributions is shown to be the set of all (R_0, R_1, R_2) triples such that

$$R_0 + R_1 + R_2 \leq I(X;Y)$$
$$R_0 + R_1 + R_2 \leq I(X;Y|U) + I(U;Z)$$
$$R_0 + R_2 \leq I(U;Z) \ .$$

Addition of feedback from both y and z to x is shown not to enlarge the capacity of the degraded broadcast channel, the Gaussian degraded broadcast channel and the deterministic broadcast channel.

I. Introduction

The broadcast channel was introduced by Cover [1] as a model for the communication network consisting of one sender and many receivers. Bergmans [2] defined the class of degraded broadcast channels as follows.

Definition 1: Z is said to be a degraded form of Y if there exists a probability transition matrix $p_3(z|y)$ such that

$$P_2(z|x) = \sum_{Y \varepsilon \mathcal{Y}} p_1(y|x) p_3(z|y) \ . \qquad (1)$$

Bergmans [2] and Gallager [3] established the capacity region for the degraded broadcast channels. As a generalization of the concept of "degradation", Kőrner and Marton [4] introduced the following two weaker partial orderings of "less noisy" and "more capable".

Definition 2: Y is said to be less noisy than Z if

$$I(U;Z) \leq I(U;Y) \qquad (2)$$

for every probability mass function of the term

$$p(u,x,y,z) = p(u)p(x|u)p(y,z|x) \ .$$

Definition 3: Y is said to be more capable than Z if

$$I(X;Z) \leq I(X;Y) \qquad (3)$$

for all probability mass function on X.

It is easy to see that,

"Degraded" => "less noisy" => "more capable".

However, it was shown by counter examples [5] that the "less noisy" definition is strictly weaker than the "degraded", and that the "more capable" is strictly weaker than the "less noisy" definition.

The capacity of the class of broadcast channels with "less noisy" ordering was shown in [4] to be similar to the degraded broadcast channel capacity region. In section II we state the capacity region for the "more capable" class.

In section III we sketch a converse as in [3] to show that if the broadcast channel was physically degraded then feedback cannot enlarge the capacity region, thus consistent with Shannon's result on the DMC with feedback [6]. Then in section IV we extend Shannon's bounds on the output entropy of the discrete time Gaussian channel to the case when feedback is added. The bounds are used to establish a converse for the Gaussian degraded broadcast channel with feedback.

Finally, we note that for the deterministic broadcast channels feedback is useless.

II. The Capacity of "More Capable" Broadcast Channels

In [4], it was proved that the channel Y is more capable than Z iff every ε-code for the channel Z is an ε-code for channel Y. Informally, this means that any independent information sent to channel Z can be decoded successfully by the decoder of channel Y. In other words, the private message to Z can always be considered as a common message to both Y and Z without affecting the capacity region.

The following theorem states the capacity of the broadcast channels with the "more capable" restriction.

Theorem 1 (Capacity region): Let $(\mathcal{X}, P(Y,Z|X), \mathcal{Y} \times \mathcal{Z})$ be a BC channel with condition (3) holding, and let U be an arbitrary random variable with cardinality $\leq ||X|| + 2$, then the capacity region C is given by

$$C = \Big\{ (R_0, R_1, R_2) : R_0 + R_1 + R_2 \leq I(X;Y)$$
$$R_0 + R_1 + R_2 \leq I(X;Y|U) + I(U;Z)$$
$$R_0 + R_2 \leq I(U;Z), \ P \varepsilon \mathcal{P} \Big\} \ , \qquad (4)$$

where $\mathcal{P}$ is the set of all probability mass functions of the form

$$p(u,x,y,z) = p(u)p(x|u)p(y,z|x) \ . \qquad (5)$$

From Figure 1, we note that

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(i) The region is symmetric in R_0 and R_2 as expected.

(ii) The plane region (R_1,R_0) coincides with the degraded message sets region given in [7].

(iii) The plane region (R_0,R_2) is defined by

$$R_0 + R_2 \le I(X;Z), \qquad (6)$$

and also coincides with the region in [7] when condition (3) is imposed.

(iv) For any fixed $R_1 = r$ the plane region (R_0,R_2) is a triangle.

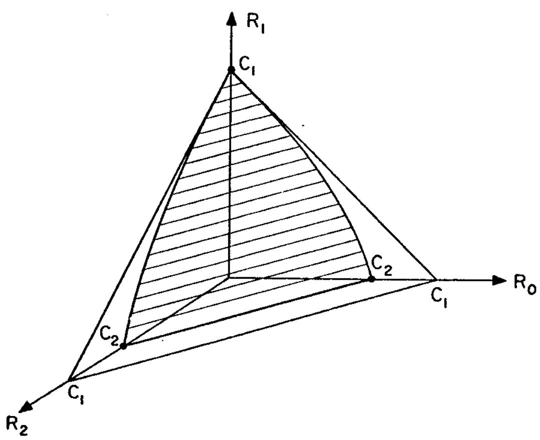

Figure 1. The Capacity Region

$$C_1 = \max_{p(x)} I(X;Z)$$

$$C_2 = \max_{p(x)} I(X;Z)$$

It is important to note that C is convex. Thus the usual convexification of the union of information regions is unnecessary. A proof of Theorem 1 can be found in [8].

III. The Capacity of the Degraded Broadcast Channel with Feedback

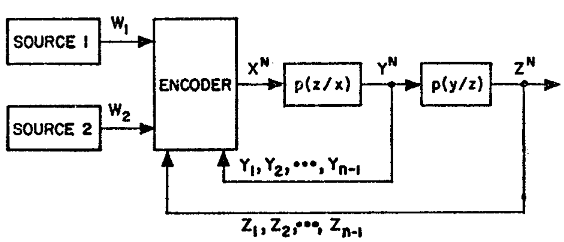

Figure 2.

Intuitively, for any broadcast channel, $Y_1,\dots,Y_{n-1},Z_1,\dots,Z_{n-1}$ are degraded forms of $X_1,\dots,X_{n-1}$, therefore one should expect that no additional information can be sent using feedback. We can prove that the intuition holds for physically degraded broadcast channels depicted in Figure 2 . The result is formalized in the following theorem.

Theorem 2 (Converse): If for some $\lambda > 0$, $\varepsilon > 0$ we have

$$R_2 + \lambda R_1 \ge C(\lambda) + \varepsilon \qquad (7)$$

where,

$$C(\lambda) = \max_{p(u)p(x|u)} \{I(U;Z) + \lambda I(X;Y|U)\}$$

then there exists $\varepsilon > 0$ such that

$$\text{Max}\left\{P_{e,1}^N, P_{e,2}^N\right\} \ge \varepsilon \quad \text{for all } N .$$

Outline of Proof.

Step 1: Fano's inequality for any codebook gives

$$N(R_2 + \lambda R_1) \le I(W_2;\underline{Z}) + \lambda I(W_1;\underline{Y}|W_2) + N\,\varepsilon_N \quad (8)$$

Step 2:

$$I(W_2;\underline{Z}) \le \sum_{n=1}^{N} I(Z_n;U_n) \qquad (9)$$

and,

$$I(W_1;\underline{Y}|W_2) = I(W_1;\underline{Y},\underline{Z}|W_2) \qquad (10)$$

$$\le \sum_{n=1}^{N} I(X_n;Y_n|U_n) \qquad (11)$$

where $U_n \triangleq (W_2,Y_1,\dots,Y_{n-1},Z_1,\dots,Z_{n-1})$ for all n .

Equation (10) is a consequence of degradation and inequality (11) results from the discrete memorylessness of the channel (see [9] for a detailed proof). We strongly believe that Theorem 2 can be extended to include all broadcast channels. However, the correlation between the outputs of the channel seem to cause some difficulty in generalizing the proof even to the stochastically degraded case.

IV. The Capacity of the Degraded Gaussian Broadcast Channel with Feedback

We first modify Shannon's entropy inequalities [10] to get upper and lower bounds on the entropy of the output of the AWGN with feedback.

Bounds on the Output Entropy.

Stam [11] and Blachman [12] have proved Shannon's statement [10] that if $\underline{X}$ and $\underline{Z}$ are two independent random vectors of length n and if

$$\underline{Y} = \underline{X} + \underline{Z} \qquad (12)$$

then

$$e^{\frac{2}{n}H(\underline{Y})} \ge e^{\frac{2}{n}H(\underline{X})} + e^{\frac{2}{n}H(\underline{Z})} \qquad (13)$$

If we let $\underline{X}$ be the input of an AWGN channel and $\underline{Z}$ the noise vector then (13) can be applied to get

$$e^{\frac{2}{n}H(\underline{Y})} \geq e^{\frac{2}{n}H(\underline{X})} + 2\pi e N , \qquad (14)$$

where N is the variance of the noise. In the feedback case, each component of the $\underline{X}$ vector can depend casually on the noise; thus, in general, $\underline{X}$ and $\underline{Z}$ are no longer independent. However, looking more closely at Blachman's proof of inequality (13) we can make the following modification.

Lemma 1 (Lower bound): Let $\underline{Z} \sim N_n(\underline{0},NI)$ and $\underline{Y} = \underline{X} + \underline{Z}$. If for all $i = 1,2,\ldots,n$, X_i and Z_i are conditionally independent given $Y^{i-1} \overset{\Delta}{=} Y_1,Y_2,\ldots,Y_{i-1}$, then

$$e^{\frac{2}{n}H(\underline{Y})} \geq e^{\frac{2}{n}\sum_{i=1}^{n} H(X_i|Y^{i-1})} + 2\pi e N \qquad (15)$$

The proof of this lemma is given in the Appendix of [13].

Next, we shall give an upper bound on the output entropy $H(\underline{Y})$ of the AWGN channel with feedback.

Lemma 2 (Upper bound): If

$$E \sum_{i=1}^{n} x_i^2 \leq nP \qquad (16)$$

then

$$H(\underline{Y}) \leq \frac{n}{2} \ln 2\pi e(P+N) \qquad (17)$$

In proving Lemma 2 we use the following useful inequality

Hadamard [14]: If $A \overset{\Delta}{=} [a_{ij}]$ is any $n \times n$ positive definite matrix, then

$$|A| \leq \prod_{i=1}^{n} a_{ii} , \qquad (18)$$

where $|A|$ denotes the determinant of A .

A Converse for the Degraded Gaussian BC Channel with Feedback.

Using Lemmas 1 and 2 and techniques similar to Bergmans [15], we can prove the following theorem for the channel in Figure 3.

Theorem 3: No point (R_1,R_2) such that for some $\alpha \varepsilon$, $[0,1]$,

$$R_i \geq C_i(\alpha) , \qquad i = 1,2$$

$$R_i = C_i(\alpha) + \delta , \text{ some } i , \delta > 0$$

is achievable, where

$$C_1(\alpha) = \frac{1}{2} \ln \left(1 + \frac{\alpha P}{N_1}\right)$$

$$C_2(\alpha) = \frac{1}{2} \ln \left(1 + \frac{\bar{\alpha} P}{N_2 + \alpha P}\right) . \qquad (19)$$

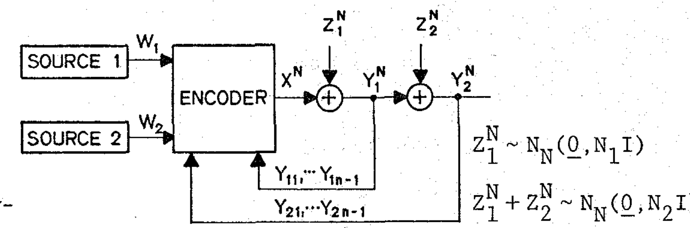

Figure 3.

Whether Theorem 3 can be extended to any Gaussian BC channel is still an open problem.

V. On the Deterministic Broadcast Channel with Feedback

We have indicated that feedback cannot increase the capacity of degraded broadcast channels. However, coding schemes using feedback may help improve error performance of codes. A class of broadcast channel for which feedback is completely useless is the so called "deterministic BC channels". An example is the Blackwell channel shown in Figure 4.

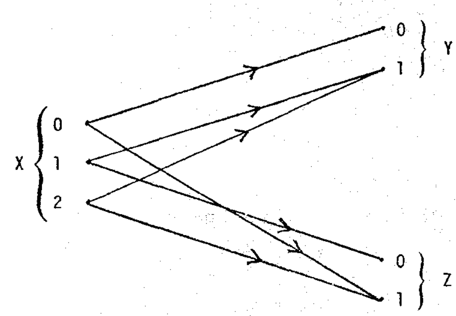

Figure 4.

Definition 4: The deterministic BC is given by three finite alphabets $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, and two functions F_1,F_2 such that $Y = F_1(x)$ and $Z = F_2(x)$. Since X determines Y and Z precisely, then there is obviously no reason to feedback any information from the outputs.

VI. Concluding Remarks

We have stated some recent results on broadcast channels. The determination of the capacity of the broadcast channel is still an open problem as well as the effect of feedback on the capacity region. The importance of the broadcast channel problem lies in the new techniques and insights developed and that will lead to a general theory of communication networks.

Acknowledgment.

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